Relativistic quantum physics

Glenn Eric Johnson
Corolla, NC
E-mail: glenn.e.johnson@gmail.com

December 16, 2024

Third Edition

Keywords: Foundations of quantum mechanics, relativistic quantum physics, constructive quantum field theory.

Contents

1 Introduction **4**

2 Realization of relativistic quantum physics **5**

3 Constructions of relativistic quantum mechanics **12**
3.1 Quantum mechanical description of state 14
3.1.1 States 15
3.1.2 The scalar product 18
3.1.3 The quantum field and VEV 20
3.1.4 Likelihoods: Born's rule 21
3.2 Axioms for the VEV of relativistic fields 22
3.3 Relativistic free fields 29
3.3.1 Free field VEV 29
3.3.2 Example $M(p)$, $S(A)$ and D 32
3.4 Nontrivial relativistic fields 34
3.4.1 Connected functions 35
3.4.2 Symmetric functions 36
3.4.3 Functions that exhibit interaction 38
3.4.4 Cluster expansion: composition of connected functions into VEV 41
3.4.5 Vacuum polarizations 47

3.5 Satisfaction of the axioms . . . 47
3.5.1 Positive definite ∘-products . . . 47
3.5.2 Positive definiteness of $\underline{\mathcal{U}}$ . . . 49
3.5.3 Poincaré invariance . . . 52
3.5.4 Cluster decomposition . . . 54
3.5.5 Regularity . . . 57
3.5.6 Summary of constructions . . . 61
3.6 Relation to free fields . . . 63
3.7 The basis function spaces . . . 63
3.7.1 The basis function spaces $\underline{\mathcal{P}}$ . . . 63
3.7.2 An alternative formulation based on $\underline{\mathcal{S}}$ . . . 65
3.8 Properties of the constructed quantum fields . . . 66
3.9 Scattering amplitudes . . . 71

4 Correspondence of quantum and classical descriptions 74
4.1 States with a quantum-classical correspondence . . . 77
4.2 The support of state descriptions . . . 81
4.3 A nonrelativistic quantum-classical correspondence . . . 84
4.4 Two body correspondence . . . 90
4.4.1 Nonrelativistic approximation . . . 91
4.4.2 Brief interval and limited acceleration approximations . . . 98
4.4.3 A functional $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ . . . 102
4.4.4 Gaussian quadratures . . . 104
4.4.5 Zeros of the derivatives of likelihood $|I(\lambda)|^2$ . . . 109
4.4.6 Energy correspondence and $-g/r$ potentials . . . 113
4.5 Extended interval propagation . . . 119

5 Technical concerns with canonical quantization 121

6 Appendices 126
6.1 The Dirac-von Neumann axioms for quantum mechanics . . . 126
6.2 Hilbert spaces and quantum mechanics . . . 128
6.2.1 Hilbert spaces . . . 128
6.2.2 Rigged Hilbert spaces . . . 129
6.2.3 The Cauchy-Schwarz-Bunyakovsky inequality . . . 130
6.2.4 Entanglement . . . 131
6.2.5 Hilbert space operators . . . 132
6.2.6 Operators in quantum mechanics . . . 133
6.2.7 Observation in Hilbert spaces . . . 134
6.2.8 Localized observables, separation and independence . . . 141

6.2.9 Tensor products of linear vector spaces 146
6.3 Translations, location operators and relativistically invariant localized states . . . 147
6.4 Location, a prototype correspondence in relativistic quantum physics 153
6.5 Inconsistency of the classical description with nature 156
6.6 The Einstein-Podolsky-Rosen paradox 159
6.7 Heisenberg uncertainty 160
6.8 Feynman-Dyson series and the constructions 165
6.9 The degenerate scalar product for free field VEV 171
6.10 Temporal evolution of function supports 173
6.11 Center-of-momentum reference frames 176
6.12 Two body classical trajectories 178
6.13 Nonrelativistic energy approximations 180
6.14 Essentially local functions 182
6.15 Example relative states 184

1 Introduction

> It seems clear that the present quantum mechanics is not in its final form. Some further changes will be needed, just about as drastic as the changes made in passing from Bohr's orbit theory to quantum mechanics. Some day a new quantum mechanics, a relativistic one, will be discovered, in which we will not have these infinities occurring at all. – Paul Dirac, in *Albert Einstein : Historical and Cultural Perspectives: The Centennial Symposium in Jerusalem*, edited by Gerald James Holton and Yehuda Elkana, 1979, p. 85.

Relativistic quantum field theory (RQFT) [7, 23, 24, 61] does not realize quantum mechanics [8, 13, 60, 64]: RQFT does not include general state descriptions with relativistically invariant transition likelihoods. If quantum mechanics suffices to describe nature, then either the constructs that realize relativistic quantum physics (RQP) have resisted discovery, or there are unnecessary considerations within RQFT that preclude physically nontrivial realization. Search for an inconsistent assertion within RQFT identifies Hermiticity[a] of the quantum field as unnecessary and presumably unrealizable for interacting fields. Hermiticity together with selection of eigenfunctions implements the quantum-classical correspondence of a canonical quantization. To exhibit only real eigenvalues, operators must be Hermitian and if, upon measurement, a system "collapses" to an eigenfunction that is a classical description, then corresponding operators must be Hermitian with eigenfunctions that are classical descriptions[b]. This conjectured quantum-classical correspondence elevates[c] classical dynamical variables to quantum mechanical operators. However, while classical dynamics must approximate "macroscopic" quantum state descriptions, those perceived as classically described, a correspondence need not be arbitrarily precise, and need not apply to every state. Indeed, no correspondence applies when indistinguishability, entanglement, and particle production or annihilation are exhibited. Such characterteristics are not classical. The assertion that field operators canonically quantize a

[a] The Hermiticity of Hilbert space operators, $A = A^*$, is a technical assumption stronger than necessary to realize the observed correspondences of classical with quantum dynamics. Hermitian fields satisfy $\Phi(\underline{f}) = \Phi(\underline{f})^*$ on a common dense domain. Fields $\Phi(\underline{f})$ consist of N_c component fields $\Phi(x)_\kappa$ defined for $\underline{f} = (f_1(x), f_2(x), \dots f_{N_c}(x))$, $x \in \mathbb{R}^4$, by $\Phi(\underline{f}) := \sum_\kappa \Phi(f_\kappa)_\kappa$. Free fields are Hermitian, $\Phi(\underline{f})^* = \Phi(\underline{f}^*)$ for "real" function sequences $\underline{f} = \underline{f}^*$. The $*$-dual function sequences $\underline{f}^*$ include an $N_c \times N_c$ linear transformation D determined by a representation of the Lorentz group (8). However, Hermiticity does not generalize from free to the physically nontrivial fields.

[b] Classical description for the observable that corresponds with the operator. The quantum state descriptions include no classical state descriptions: the Heisenberg uncertainty principle depicts that both location and momentum can not be specified in a quantum description.

[c] Here, a canonical quantization is also designated an *elevation*. Considering the quantum state descriptions as functions over domains determined by the corresponding classical dynamical variables, the canonical quantization of a classical dynamical variable is the Hilbert space operator implemented as multiplication of the state describing function by an argument, e.g., $X\psi(x) = x\psi(x)$. Then, eigenfunctions of elevations are either Dirac or Kronecker delta functions.

classical field is an extrapolation of the "elevation of c-number to q-number" concept from early, nonrelativistic quantum mechanics [13, 60].

Constructions are provided in these notes to demonstrate that physically nontrivial relativistic quantum physics is realizable. These realizations lack Hermitian fields when interaction is exhibited. Nevertheless, with the exception of the Hermiticity of fields, constructions satisfy the established principles of quantum mechanics and relativity [13, 61]. Justified by the lack of Hermiticity, the constructions, section 3 and [31, 33, 35, 37], violate the Jost lemma. The Jost lemma applies if fields are densely defined Hermitian operators and provides that a Pauli-Jordan function can be the two-point vacuum expectation values (VEV) only for free quantum fields. Pauli-Jordan function two-point VEV are included by the physically nontrivial examples.

Discussion emphasizes foundation principles for relativistic quantum physics, establishment of example realizations, and a physical understanding of the development. Associated phenomenology is relatively unexplored other than to establish connections with Feyman series and classical dynamics. Primary references are Borchers' [10] and Wightman's [56, 64] developments of quantum field theory, Bogolubov, Logunov and Todorov's review [9], and mathematical background includes [3, 12, 19, 20, 21, 24, 27, 40, 46, 48, 60].

2 Realization of relativistic quantum physics

To satisfy observation, appropriate classical and quantum state descriptions must correspond but a canonical quantization is not necessary, and for relativistic location, is not possible, [43, 66] and appendix 6.4. A technically revised quantum-classical correspondence is realizable. Relativistic location provides an archetype for realizable quantum-classical correspondences. The canonical quantization of classical location is not Hermitian in relativistic physics: delta functions are not eigenfunctions of a Hermitian operator in relativistic physics.[d] Elevations X_1, X_2, X_3 of location $\mathbf{x} \in \mathbb{R}^3$,

$$X_\nu := -i\frac{d}{d\mathbf{p}_\nu},$$

[d] Eigenfunctions of a Hermitian operator are necessarily orthogonal if they have distinct eigenvalues [3]. Dirac delta functions are not orthogonal for a relativistic scalar product.

$$\int dxdy\ \Delta^+(x-y)\delta(x-x_o)\delta(y-y_o) = \Delta^+(x_o-y_o) \neq 0$$

for $x_o \neq y_o$ with $\Delta^+(x)$ a Källén-Lehmann form [9, 54]. There are elevations of the momentum operators,

$$\int dxdy\ \Delta^+(x-y)e^{-ip_1x}e^{ip_2y} = (2\pi)^4\delta(p_1-p_2)\theta(E_2)\delta(p_2^2-m^2) = 0$$

if $p_1 \neq p_2$ for the example of the Pauli-Jordan function. The e^{ipx} are inverse Fourier transforms of Dirac delta functions over momenta.

are distinct from the Hermitian relativistic location operators[e] [43]

$$\hat{X}_\nu := -i\,\omega^{1/2}\frac{d}{d\mathbf{p}_\nu}\,\omega^{-1/2}. \tag{1}$$

The operators X_ν and $\hat{X}_\nu$ are expressed in the Fourier transform, momentum domain and $\nu = 1, 2, 3$.[f] That the canonical quantization of location is not a Hermitian operator illustrates a "localization problem" in RQFT. Nevertheless, the elevated location operators X_ν conditionally approximate the Hermitian operators $\hat{X}_\nu$ in the sense that the point support of eigenfunctions of X_ν are representatives for the support of the eigenfunctions of the Hermitian location operators $\hat{X}_\nu$, appendices 6.3 and 6.4. The non-Hermitian X_ν approximate the Hermitian $\hat{X}_\nu$ if applied to appropriate state descriptions. That is,

$$X_\nu \approx \hat{X}_\nu$$

when applied to functions dominantly supported on nonrelativistic momenta, those with

$$\hbar^2\mathbf{p}^2 \ll (mc)^2$$

within the dominant support.[g] Relativistic location demonstrates that the presumed quantum-classical correspondence of a canonical quantization imposes unrealizable constraints on relativistic quantum physics.

In relativistic quantum mechanics, nature is described by sequences of complex-valued functions over three dimensional space and the evolution of these functions is parametrized by time

[e]The Newton-Wigner location operators $\hat{X}_\nu$ canonically commute with the Hermitian elevations of momenta and is Hermitian for the relativistic free field scalar product, appendix 6.3. From the Baker-Campbell-Hausdorff relations, momentum generates spatial translations.

[f]Spacetime vectors $x := (x_0, \mathbf{x})$ with $x_0 = ct$ and spatial vectors $\mathbf{x} := x, y, z \in \mathbb{R}^3$ are lengths, energy-momenta are wavenumbers designated $p := (p_0, \mathbf{p})$ with momentum vectors $\mathbf{P} = \hbar\mathbf{p} \in \mathbb{R}^3$, section 3.1.1. Energies are $E := \hbar c p_0$ and $\hbar$ is Planck's constant h divided by 2π. c is the speed of light. Momentum vectors $\mathbf{p}$ have components $\mathbf{p}_\nu$, $\nu = 1, 2, 3$ and $\omega = \omega(\mathbf{p})$ is the wavenumber proportional to an energy on the mass m shell.

$$\omega := \sqrt{m^2c^2/\hbar^2 + \mathbf{p}^2}.$$

[g]For example, in the $\mathcal{L}^2(\mathbb{R}^3)$ norm,

$$\frac{\|(X_\nu - \hat{X}_\nu)\psi\|}{\|\psi\|} \leq \frac{\lambda_c^2}{\sqrt{8}\,\alpha} \ll \lambda_c$$

if $\lambda_c \ll \alpha$ with α characterizing the extent of the spatial support and $\hbar/\alpha$ characterizing the extent of the momentum support of $\psi(\mathbf{x}) = \exp(-\mathbf{x}^2/(2\alpha^2))$. λ_c is the Compton wavelength (14) for the finite mass m particle. The condition $\lambda_c \ll \alpha$ provides that the dominant support of $\psi(\mathbf{x})$ is nonrelativistic. On appropriate states, the expected values of X_ν and $\hat{X}_\nu$ are nearly equal neglecting location differences small with respect to the Compton wavelength.

[9, 10]. The sequences of state describing functions over a 3+1 spacetime

$$\underline{f} := (f_0, f_1(x_1)_1, \dots f_1(x_1)_{\kappa_1}, \dots f_2(x_1, x_2)_{\kappa_1,\kappa_2}, \dots)$$

label elements $|\underline{f}\rangle$ of a rigged (equipped) Hilbert space $\mathbf{H}_{\mathcal{P}}$, [9, 10, 31, 33, 37, 56], section 3 and appendix 6.2. Each $x_j \in \mathbb{R}^4$, $j, \kappa_j \in \mathbb{N}$, $\kappa_j \in \{1, N_c\}$ with N_c the number of field components.

$$|\underline{f}\rangle := \sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \Omega\rangle \tag{2}$$

with each of the $(N_c)^n$ functions $f_n((x)_n)_{(\kappa)_n} \in \mathbf{H}_{\mathcal{P}}$. $f_0 \in \mathbb{C}$ and

$$(x)_n := x_1, x_2 \dots x_n \in \mathbb{R}^{4n}.$$

Spaces of functions $\mathcal{P}(\mathbb{R}^{4n})$ are subspaces of the space of Schwartz tempered functions $\mathcal{S}(\mathbb{R}^{4n})$ [19] and there is a subspace $\mathcal{P}_{(\kappa)_n}(\mathbb{R}^{4n})$ determined for each

$$(\kappa)_n = \kappa_1, \kappa_2 \dots \kappa_n \in \mathbb{N}^n$$

that has a distinct sequence of masses $m_{\kappa_1}, m_{\kappa_2}, \dots m_{\kappa_n}$, section 3.7. Test function sequences $\underline{f} \in \underline{\mathcal{P}}$ are dense in $\mathbf{H}_{\mathcal{P}}$. $\prod_k \Phi(x_k)_{\kappa_k}$ is a product of quantum fields implemented as a multiplication $\underline{f} \times \underline{g}$ of function sequences, [10] and section 3.1.3.

$$\sum_{n=0}^{\infty} \sum_{(\kappa)_n} \int d(x)_n \; f_n((x)_n)_{(\kappa)_n} | \prod_{k=1}^{n} \Phi(x_k)_{\kappa_k} \underline{g}\rangle := |\underline{f} \times \underline{g}\rangle. \tag{3}$$

The $\times$-multiplication of function sequences is

$$\underline{f} \times \underline{g} := (f_0 g_0, \dots, \sum_{\ell=0}^{n} f_\ell(x_1, \dots x_\ell)_{(\kappa)_{1,\ell}} \, g_{n-\ell}(x_{\ell+1}, \dots x_n)_{(\kappa)_{\ell+1,n}}, \dots). \tag{4}$$

The sequence

$$\Omega = (1, 0, 0 \dots)$$

describes the vacuum and Ω is the identity for the $\times$-multiplication of sequences. In (2), the quantum fields map

$$\Omega \in \underline{\mathcal{P}} \quad \mapsto \quad \underline{f} \in \underline{\mathcal{P}}$$

from $\underline{f} \times \Omega = \underline{f}$. Notation is discussed further in section 3.1.

The vacuum expectation values (VEV) of the quantum fields

$$\langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle \in \mathcal{S}'(\mathbb{R}^{4n}) \tag{5}$$

are generalized functions (distributions) in the dual to Schwartz tempered test functions $\mathcal{S}(\mathbb{R}^{4n})$. These VEV together with specification of the function sequences $\underline{\mathcal{P}} \subseteq \underline{\mathcal{S}}$ determine the Hilbert space realization for RQP. The scalar product in $\mathbf{H}_{\mathcal{P}}$ is

$$\begin{aligned}\langle \underline{f}|\underline{g}\rangle := \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, \langle \Phi(x_1)_{\kappa_1} \dots \Phi(x_n)_{\kappa_n} \Omega | \Phi(x_{n+1})_{\kappa_{n+1}} \dots \Phi(x_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ \times \overline{f_n}(x_1, \dots x_n)_{\kappa_1 \dots \kappa_n} \, g_m(x_{n+1}, \dots x_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}}\end{aligned} \tag{6}$$

with formal summation notation for the generalized functions [19]. From Born's rule, VEV determine likelihoods. The $|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$ are equivalence classes within the Hilbert space-norm completion of terminating function sequences $\underline{f} \in \underline{\mathcal{P}}$ [9, 10, 12, 56]. $\mathbf{H}_{\mathcal{P}}$ includes elements described by generalized functions with point support over time, functions used by Lehmann, Symanzik and Zimmermann (LSZ) to describe scattering, [9] and section 3.9. Here, vacuum expectation values (5) of quantum fields $\Phi(x)_\kappa$ satisfy the principles of quantum mechanics and relativity, described in axioms A.1-7 in section 3.2, without imposition of unnecessary constraints from canonical quantization. The axioms revise the Wightman axioms [9, 10, 56, 62] with additional considerations from RQFT [61].

Consistently with canonical quantization's "elevation of c-number to q-number" correspondence, satisfaction of the Wightman [56, 62] axioms provides that quantum fields are densely defined Hermitian operators. The Wightman axioms require that the scalar product (6) simplifies to

$$\begin{aligned}\langle \underline{f}|\underline{g}\rangle &= \underline{W}(\underline{f}^* \times \underline{g}) \\ &:= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \, \langle \Omega | \Phi(x_1)_{\kappa_1} \dots \Phi(x_{n+m})_{\kappa_{n+m}} \Omega \rangle \\ &\qquad \times f_n^*(x_1, \dots x_n)_{\kappa_1 \dots \kappa_n} \, g_m(x_{n+1}, \dots x_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}}\end{aligned} \tag{7}$$

with $\underline{f} \times \underline{g}$ the same product of function sequences as in the definition (3) of field [10]. The $*$-dual sequence $\underline{f}^* \in \underline{\mathcal{S}}$ is the result of an argument order reversal, complex conjugation and a linear transformation D determined by representation of the Lorentz group.

$$\widetilde{f_n^*}((p)_n)_{(\kappa)_n} := (D^T \cdot)_n \overline{\tilde{f}_n}(-p_n, \dots - p_1)_{\kappa_n \dots \kappa_1} \tag{8}$$

in matrix notation with $(D^T)_{ij} := D_{ji} \in \mathbb{C}$, $\tilde{f}(p)$ is the Fourier transform (16) of $f(x)$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$ and

$$(D\cdot)_n V_{(\kappa)_{n+m}} := \sum_{\ell_1=1}^{N_c} \sum_{\ell_2=1}^{N_c} \dots \sum_{\ell_n=1}^{N_c} D_{\kappa_1 \ell_1} D_{\kappa_2 \ell_2} \dots D_{\kappa_n \ell_n} V_{\ell_1 \dots \ell_n \kappa_{n+1} \dots \kappa_{n+m}}. \tag{9}$$

Example D are provided in sections 3.1.2, 3.3.1 and 3.3.2. $\underline{W}(\underline{f})$ is the Wightman functional [9, 10, 56, 62]. However, despite concerted efforts, no physically nontrivial realization for $\underline{W}$

have been discovered and (7) is not satisfied by the constructions that exhibit interaction. The property of VEV that satisfy (7) is designated *formal Hermiticity* below. To construct VEV that exhibit interaction, the formal Hermiticity constraint is abandoned: physically trivial free field VEV satisfy formal Hermiticity. The more general scalar product (6) is adopted for axiom A.1-7 compliance. As a consequence, and consistently with the Haag (Haag-Hall-Wightman-Greenberg) theorem [9, 56, 64], unitary similarity of interacting and free field operators is abandoned. Although, as discussed below, RQFT developments persist as approximations to the constructions.

Motivated by several concerns, quantum field operators that are not Hermitian operators and revised quantum-classical correspondences are studied in this note. Concerns include:

1. Hermitian elevations may be inconsistent with relativity. Relativistic invariance of likelihoods implies that functions with point support are not eigenfunctions of a Hermitian operator in RQP: due to relativity, there is no exact correspondence of location as a classical dynamical variable with an argument of state describing functions [43, 66]. However, the physical relevance of RQFT suggests that an "approximate elevation" of location suffices. This approximation applies conditionally

2. an exact, "collapse to an eigenfunction" that provides a classical description is stronger than required by our observations. Conditional and approximate correspondences of classical and quantum state descriptions suffice. The stronger assertion underlies the elevation of classical dynamical variable to densely defined Hermitian operator. It is neither possible to prepare all of a dense set of natural states nor to verify that, for example, location corresponds precisely to one real number. If the eigenfunctions of corresponding Hermitian operators are accurately represented to great likelihood by classical dynamical variables in "macroscopic" instances, then a quantum-classical correspondence indiscernible from exact is established

3. quantum fields do not necessarily have any eigenfunctions, section 3.1.3

4. a physically equivalent development for the established free quantum field lacks Hermitian field operators [31]. This alternative construction demonstrates that Hermitian fields are not necessary to realize relativistic quantum physics

5. there are realizations of RQP consistent with the principles of quantum mechanics and relativity that lack Hermitian fields, [31, 33, 35, 37] and section 3. Constructed scattering amplitudes approximate Feynman series scattering amplitudes, section 3.9 and [31, 35]. For the constructions with a single finite mass elementary particle, a short range, Yukawa-like equivalent potential suggestive of nuclear forces is associated with scattered states in first Born approximation [31, 33]; and long range $-g/r$ pair potentials suggestive of Newtonian gravity or electrostatics are associated with the evolution of significantly separated,

classical body-like concentrations in the support of states in nonrelativistic approximations, section 4. Generally, one quantum dynamical construction exhibits multiple classical correspondences determined by properties of the state descriptions and fidelity of the approximation.

Here, non-Hermitian "approximately elevating" quantum fields are considered. This departure from earlier constructive efforts [2, 4, 9, 30, 39, 41, 56] results in the physically nontrivial realizations of RQP. A revised understanding of the quantum description of relativistic physics, sections 3.1 and 4, enables the realizations of RQP with quantum fields that only approximate canonical quantizations of classical fields. In the revised understanding, the quantum-classical correspondence is established by representing the support of quantum state describing functions with classical dynamical variables. And, classical dynamical variables necessarily provide accurate representatives for the quantum description of state only when support is isolated and well represented by one location and one momentum. State descriptions with isolated support well represented by one location and one momentum are designated here as "macroscopic." This approximate and conditional correspondence of classical and quantum state descriptions substitutes for the elevations conjectured in canonical quantization. Elevations establish correspondences of classical dynamic variables with the eigenfunctions of hypothesized densely defined Hermitian operators; this exacting correspondence is contradicted by relativistic location. The revision replaces this curious extrapolation with a more physically justifiable conditional and approximate correspondence of state descriptions.

Physically nontrivial VEV include multiple argument connected contributions.

$$\langle\tilde{\Phi}(p_1)\dots\tilde{\Phi}(p_k)\Omega|\dots\tilde{\Phi}(p_n)\Omega\rangle = \dots + c_n\delta(p_1+p_2+\dots+p_n)\prod_{j=1}^{n}\delta(p_j^2-m^2c^2/\hbar^2) \tag{10}$$

in an example construction of a single neutral scalar field $\Phi(x)$, section 3.4 and [31, 33]. In (10), $N_c = 1$, $n \geq 4$, $n-2 \geq k \geq 2$, the contributions of the less than n-point connected functions are understood, and $\tilde{\Phi}(p)$ designates the Fourier transform (17) of $\Phi(x)$. VEV are cluster expansions of connected functions, section 3.4.4. The selection of basis function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ limits the states in $\mathbf{H}_{\mathcal{P}}$ to positive energies. The basis function spaces $\underline{\mathcal{P}}$ are limited to functions with Fourier transforms that vanish on appropriate negative energy mass shells: field component κ_j is associated with mass m_{κ_j}, section 3.7.

$$\delta(p_j^2-m_{\kappa_j}^2c^2/\hbar^2)\theta(-E_j)\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = 0 \tag{11}$$

if $\varphi_n \in \mathcal{P}(\mathbb{R}^{4n})$ with $1 \leq j \leq n$. Except for an unobservable[h] phase difference between forward and scattered contributions, the plane wave scattering amplitudes resulting from VEV such as

[h] Unobservable in the scattering limit, and RQFT methods do not provide estimates for finite transition intervals.

(10) coincide with first order terms from a Feynman-Dyson series [31, 33] with, in some examples, corrections at very relativistic exchange momenta (small distances) [35]. For the neutral scalar field example (10), the scattering cross sections coincide with the first order contributions from $:P(\Phi)_4:$ interactions. The phase difference between (10) and the Feynman-Dyson series is necessary to the nonnegativity of the scalar product (6). Due to introduction of the support constraint, the algebra of function sequences $\underline{\mathcal{P}}$ is not $*$-involutive and as a consequence, the fields in (5) are not Hermitian Hilbert space operators if they exhibit interaction. With interaction, the adjoints of the field operators are not Hilbert space operators, section 3.8. The support constraint also implies that there are no states strictly limited to bounded spatial volumes described by functions in $\mathbf{H}_{\mathcal{P}}$ [33]. Considered as functions over spacetime, functions within $\underline{\mathcal{P}}$ do not vanish within any finite spatial volume unless the function is identically zero: such functions are designated *anti-local* [53]. Nevertheless, comparing summations over equal finite volumes, anti-local functions include functions arbitrarily dominantly supported within one finite spatial volume: such functions are designated *essentially localized* here, appendix 6.14. The physically significant support of functions in $\underline{\mathcal{P}}$ may be local, section 4.1. For the VEV (10) to be generalized functions with massless particles, at least 3+1 dimensional spacetime is necessary, section 3.5.5.

Equivalently, with VEV modified from formally Hermitian forms such as (10), the basis function space includes all sequences of the Schwartz tempered functions $\underline{\mathcal{S}}$, section 3.7.2. $\underline{\mathcal{S}}$ includes dense sets of real functions with bounded support. If based upon $\underline{\mathcal{S}}$, then the VEV for the neutral scalar field example (10) are the first order contributions from a Feynman-Dyson series with a phase difference, appendix 6.8 and [31, 35],

$$\langle\tilde{\Phi}(p_1)\dots\tilde{\Phi}(p_k)\Omega|\dots\tilde{\Phi}(p_n)\Omega\rangle = \dots + c_n\delta(p_1+p_2+\dots+p_n)\prod_{j=1}^{k}\theta(-E_j)\delta(p_j^2-m^2c^2/\hbar^2) \times\prod_{\ell=k+1}^{n}\theta(E_\ell)\delta(p_\ell^2-m^2c^2/\hbar^2).$$

These scalar field Feynman-Dyson series VEV truncated at first order satisfy A.1-7 but do not satisfy the Wightman axioms [31]. These VEV do not satisfy formal Hermiticity. While the algebra of basis test functions $\underline{\mathcal{S}}$ is $*$-involutive, the variation of the VEV with k precludes Hermiticity of the field operators in the VEV (5), section 3.8. The connection of developments based on the energy support limited $\underline{\mathcal{P}}$ and on $\underline{\mathcal{S}}$ is the equivalence

$$\frac{\omega_j \pm p_{j0}}{2\omega_j} = \theta(\pm E_j) \tag{12}$$

as multipliers of generalized functions supported solely on mass shells. The lefthand side is a multiplier function [20] applicable to test functions and defines the function spaces $\underline{\mathcal{P}}$ from $\underline{\mathcal{S}}$, section 3.7, while the Heaviside functions $\theta(E)$ evidently limit the support of states to positive

energies. The wavenumber

$$\omega_j := \omega(\mathbf{p}_j) := \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2} \tag{13}$$

and the reduced Compton wavelength for the mass m_{κ_j} associated with a field component $\Phi(x_j)_{\kappa_j}$ is

$$\lambda_{cj} := \frac{\hbar}{m_{\kappa_j} c}. \tag{14}$$

The alternative RQP constructions present the same puzzlement as the Reeh-Schlieder theorem from RQFT [45], section 3.7.2: although functions with local support can be considered to label states in relativistic quantum physics, the spatial support of quantum fields is global.

Realization of relativistic quantum physics includes: a physical understanding of the state descriptions $|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$ (2); the VEV (5) that determine the scalar product (6); and the basis space of test function sequences $\underline{f} \in \underline{\mathcal{P}}$ (11). Revisions to the Wightman axioms are the prospective axioms for relativistic quantum physics. The physical conditions of the Wightman axioms are preserved but the condition that implies densely defined Hermitian fields, formal Hermiticity, is removed. The newly constructed VEV in section 3.4 satisfy axioms that include additional revisions from [31, 33, 37], section 3.2. A stronger cluster decomposition property than the uniqueness of the vacuum condition used in [31, 33, 37] provides that truncated functions [9] are connected functions and this stronger condition is included as an axiom. The stronger cluster decomposition condition: provides a replacement for formal Hermiticity in demonstrations that the quantum fields (3) are (non-Hermitian) Hilbert space operators; implies that states with sufficiently isolated and space-like separated support are described by free particles; provides the essential independence of the local observables of non-entangled, spatially distant bodies; and implies a unique vacuum.

3 Constructions of relativistic quantum mechanics

> In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved. All depends, then, on finding out these easier problems, and on solving them by means of devices as perfect as possible and of concepts capable of generalization. – David Hilbert, in *Bulletin of the American Mathematical Society* 8 (1902), 437-479.

In current understanding, the natural world is described by quantum mechanics. In quantum mechanics, bodies are described by complex-valued functions over three dimensional space. The evolution of these functions is parametrized by time. The resulting functions over a 3+1 spacetime label elements of the rigged (equipped) Hilbert space introduced in section 2. The

evolution of states, likelihoods of observations, quantum-classical correspondences and properties of the field derive from the choice of vacuum expectation values (VEV) for products of the field and the choice of basis function spaces [9, 10, 31, 33, 37, 56]. The VEV specify the scalar product $\langle \underline{f} | \underline{g} \rangle$ of Hilbert space elements $|\underline{f}\rangle, |\underline{g}\rangle \in \mathbf{H}_{\mathcal{P}}$ including limits of elements from the basis space of function sequences $\underline{\mathcal{P}}$.

Example VEV (5) and spaces of function sequences $\underline{\mathcal{P}}$ (11) that realize relativistic quantum physics are constructed in this section. The VEV generalize the single neutral scalar field example (10) introduced in section 2. The constructions exhibit properties of nontrivial relativistic physics with multiple component quantum fields $\Phi(x)_\kappa$. In section 3.1, rigged Hilbert space realizations of quantum physics are discussed, and prospective axioms characterizing the constructions are presented in section 3.2. The VEV of the relativistic free fields are discussed in the current context in section 3.3.1 and appendix 6.9. The VEV are constructed as cluster expansions of symmetric, connected functions. Symmetric, connected functions are constructed and the cluster expansion defined in section 3.4. Satisfaction of the axioms is demonstrated in sections 3.5.1-3.5.5. In section 3.7, the basis spaces $\underline{\mathcal{P}}$ that provide nonnegativity of energies (11) are presented, and the equivalence with constructions based on $\underline{\mathcal{S}}$ demonstrated. The construction is summarized and the Hamiltonian operator evaluated in section 3.5.6. The constructions are developed further in sections 3.8-3.9, section 4 and the appendices. The scattering amplitudes are illustrated in section 3.9.

In the designations of appendix 6.1, quantum mechanics is described by Dirac-von Neumann axioms I-III. Dirac-von Neumann axioms IV and V describe the canonical quantization of nonrelativistic physics. In the constructions, satisfaction of axioms I-III is maintained in A.1-7, IV remains true for location and momentum but not for fields, and V does not apply. In relativistic physics, the quantum-classical correspondence described in axioms IV and V is replaced with a more appropriate correspondence for relativistic physics. The adopted quantum-classical correspondence compares classical and quantum state descriptions. The technically relaxed correspondences derives from Erwin Schrödinger's 1926 study of the linear harmonic oscillator [49] and the generalization from Paul Ehrenfest [14, 42]: classical dynamical variables are representatives for the support of appropriate state describing functions. Appropriate state describing functions are discussed in sections 2 and 4. The most classical-like states are as nearly well represented by classical dynamical variables as possible: the Heisenberg uncertainty lower bound is satisfied. Nature is described by equivalence classes of functions and the physically significant features of these functions are Lebesgue summations over measurable subsets of $\mathbb{R}^3$. A correspondence of Lebesgue measurable subsets with points $\mathbf{x}$ or $\mathbf{p} \in \mathbb{R}^3$ is necessarily inexact. This adopted quantum-classical correspondence is suited to relativity and results in decisive revision to the mathematical development of RQP.

From (12), there are two equivalent approaches to the constuctions: constrained functions $\underline{\mathcal{P}}$ and VEV with support on negative energies; and unconstrained functions $\underline{\mathcal{S}}$ with VEV appropriately constrained to positive energies. The approach based upon the constrained functions $\underline{\mathcal{P}}$ is emphasized here.

3.1 Quantum mechanical description of state

Key considerations for realization of relativistic quantum physics include:

1. functions are the description of nature. Functions over spacetime and quantum numbers label elements of rigged Hilbert spaces, and the Fourier transforms of these functions describe momenta

2. the Hamiltonian generates time translation. To comply with relativity, the Hamiltonian must be a generator of a Hilbert space realization of the Poincaré group.

In the classical concept, observables are classical dynamical variables. The classical concept includes an observer who can describe the evolution of each identifiable body. The description of state is considered independently of observation, or perhaps more precisely, the state is considered determined and observation need not disturb that state. This geometric description with an omniscient observer is what is meant by *classical physics* in these notes. Quantum mechanics supersedes, not "quantizes," classical descriptions. A description of nature as elements of rigged Hilbert spaces manifests the wave-particle duality, and implements the discrete line spectra of atomic emissions, the quantized energy of photons, the entanglement of states necessary to consistent description of quantized conserved quantities, and the indistinguishability of similarly described bodies that results in an extensive entropy. The quantum description is fundamentally contradictory to classical concepts. Nevertheless, when the dominant supports of state describing functions are well represented by a single location and momentum, and isolated, the description provided by the state describing function is classical body-like. For these particular state descriptions, there is a close correspondence of quantum and classical descriptions, section 4. More generally, description is inherently quantum mechanical. For widely supported states, or significantly overlapping descriptions, or when entanglement applies, or when non-commuting observables are considered, or on small spatial scales, or for relativistic collisions, classical description is contradicted. For finite mass particles, the scale for "small" is generally set by the Compton wavelength (14). Trajectories for distinguishable bodies apply in quantum mechanics only while a correspondence with classical bodies applies. The Heisenberg uncertainty principle is characteristic of the description of nature. The description of state precludes classical description; states of nature are never classically described despite the familiar and useful classical approximations. The instrinsic understanding of quantum mechanics is the relative state (Everett-Wheeler-Graham) interpretation, [11] and appendix 6.2.7. The resulting description of observation explains how common perceptions of nature differ from their description, and resolves the Einstein-Podolosky-Rosen (EPR) [16], Schrödinger's cat [51] and Wigner's friend [65] paradoxes associated with earlier understandings of quantum mechanics. Quantum mechanics contradicts classical description and in this sense the concepts are disconcerting. But, quantum mechanics is necessitated by consistency with nature. The classical view is not supported by our broadened observations of nature.

The description of nature by functions is the result of an extraordinary effort to rectify observation with prediction begun in the early twentieth century. The initial effort is notably due to Max Planck, Albert Einstein, Max Born, Werner Heisenberg, Erwin Schrödinger, Paul Dirac, Wolfgang Pauli, Pascual Jordan, Enrico Fermi, Neils Bohr, Hermann Weyl, John von Neumann and Eugene Wigner, and efforts to understand the resulting formalism of quantum mechanics are notably due to Hugh Everett III, Bryce DeWitt and J.A. Wheeler. The correspondence of classical and quantum descriptions of nature is discused further in section 4. The classical description is a perception facilitated by often excellent approximation.

3.1.1 States

The sequences of functions that describe nature are elements of a rigged Hilbert space. These functions can depict the vacuum, a single body, multiple bodies, waves or unparticle-like [25] states. Perceptions of these states by observers and the temporal evolution of states are the concerns of mechanics. This Hilbert space description fulfills essential properties of nature [13, 60]. Hilbert spaces are discussed in appendix 6.2.

The elements $|\underline{f}\rangle$ of the equipped or *rigged Hilbert spaces* of interest $\mathbf{H}_{\mathcal{P}}$ are within the completion of elements described by terminating sequences of multiple argument functions from the basis function spaces $\underline{\mathcal{P}}$ [9, 10, 12, 56].

$$|\underline{f}\rangle \in \mathbf{H}_{\mathcal{P}}$$

is described by any one of an equivalence class of function sequences

$$\underline{f} := (f_0, f_1(x_1)_1, f_1(x_1)_2, \dots f_1(x_1)_{N_c}, \dots f_n((x)_n)_{(\kappa)_n}, \dots) \tag{15}$$

with

1. $f_0 \in \mathbb{C}$ the component of the state in the vacuum state characterized by the sequence

   $$\Omega := (1, 0, 0 \dots)$$

2. N_c functions $f_1(x)_\kappa$ that describe single particle states with masses m_κ, $\kappa \in \{1, 2, \dots N_c\}$

3. for $n \geq 2$, there are $(N_c)^n$ functions $f_n((x)_n)_{(\kappa)_n}$,

   $$f_n((x)_n)_{1\dots11}, f_n((x)_n)_{1\dots12}, \dots f_n((x)_n)_{N_cN_c\dots N_c}$$

   in the n-argument subspace. Generally, $f_n((x)_n)_{(\kappa)_n}$ is not interpretable as describing any determined number or species of particles. Nevertheless, the functions $f_n((x)_n)_{(\kappa)_n}$

describe n-particles in the absence of significant interaction, for example, if the support[i] of each argument is distantly space-like isolated from the support of any other state describing function argument, or the momentum support is nonrelativistic.

Each natural number $\kappa_j \in \{1, 2, \ldots N_c\}$ labels one of N_c field components. The basis function spaces $\underline{\mathcal{P}}$ are subspaces of the space of terminating sequences of Schwartz tempered functions $\underline{\mathcal{S}}$. $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ provides that $\underline{\mathcal{S}}' \subseteq \underline{\mathcal{P}}'$ for the duals [20] but for the constructions, the constructed VEV lie within $\underline{\mathcal{S}}'$.

The labels on arguments do not identify particular bodies. Gibb's paradox is resolved by the indistinguishability of similarly described bodies. If the spatial supports of each argument of the state describing functions $f_n((x)_n)_{(\kappa)_n}$ has the same mass, charge and polarization, and overlap, the results of observation of a body are not reliably associated with the support of a single argument. Unless the support of an argument of the state describing function is isolated to great likelihood, a distinguished particle can not be reliably associated with the support of a state describing function argument. Isolation of support enables association of a particular particle description and volume of space, with the volume of space and support of an argument. Indistinguishability is implemented in symmetry of the squared magnitude of the scalar product with interchange of argument labels. If $x_j - x_k$ is space-like in the support of $f_n((x)_n)_{(\kappa)_n}$, then the state described by $f(\ldots x_j, x_k, \ldots)_{(\kappa)_n}$ is equivalent to the state described by either $\pm f(\ldots x_k, x_j, \ldots)_{(\kappa)_n}$.[j] A quantum-classical correspondence is conditioned on the properties of a state description.

In relativistic physics when interaction is manifest, particle number is not conserved and states with different numbers of particles are generally not orthogonal in the constructed scalar products. With interaction and if $n \geq 2$, an n-argument function is not generally associated with any fixed number of bodies $k \geq 2$. But even with interaction, there are states well represented as n classical particles. For free field VEV and functions $f_n \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4n})$, an n-argument function describes an n-particle state [31]. But, more generally, since scalar products of functions with different numbers of arguments do not vanish, there are nonzero likelihoods of observing $k \neq n$ particles and distinct particle species for a state described by an n-argument function. With the exceptions of the vacuum ($n = 0$) and functions of a single argument ($n = 1$), functions with n arguments are generally not orthogonal to $k \neq n$ argument functions for the physically nontrivial scalar products.

[i] The support of a function is usually defined as the set of points in the domain where the function is not zero. Here, the dominant support of a function refers to finite volumes within the domain that include the dominant contribution to a norm of the function. Norms include the Hilbert space and $\mathcal{L}^2$ norms. The norms are truncated to finite volumes to assess dominant support. Then, the support of a function includes Lebesgue measureable sets of points in the domain where the finite volume contribution to the norm is not negligible. Comparing equal finite volumes, a function has more support in volumes with greater norm. Localized designates that the dominant support is primarily within one finite volume. The radius of an enclosing sphere characterizes the size of the support. Relatively negligible norms within finite volumes are taken as zero to apply the usual definition of function support. Generalized functions exhibit both regular and singular support [19].

[j] For brevity, the possibility of *parastatistics* [9] is not included here.

To describe momentum, functions in the basis function spaces $\underline{\mathcal{P}}$ are required to have Fourier transforms. $\tilde{f}_n((p)_n)$ denotes the Fourier transform of $f_n((x)_n)$. The Fourier transform adopted here applies in four dimensional spacetime and is the evident multiple argument extension of

$$\tilde{\psi}(p) := \int \frac{dx}{(2\pi)^2}\, e^{-ipx}\psi(x) \tag{16}$$

using wavenumber p, energy-momentum $P := \hbar p$, $E = cP_0$ and the Lorentz invariants $px = p_0ct - \mathbf{p}\cdot\mathbf{x}$ and spacetime volume element $dx = dx_0dx_1dx_2dx_3$. $\hbar$ is Planck's constant h divided by 2π. To describe relativity, spacetime coordinates in four dimensions are designated $x := (ct, \mathbf{x})$ and energy-momentum vectors are $p := (E/(\hbar c), \mathbf{p})$ and $\mathbf{p} := (\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z)$. c is the speed of light and px is without units. $Px = \hbar px := Et - \mathbf{P}\cdot\mathbf{x}$. $x, p \in \mathbb{R}^4$ are Lorentz four-vectors and $\mathbf{x}, \mathbf{p} \in \mathbb{R}^3$ are three dimensional Euclidean vectors. $x^2 := (ct)^2 - \mathbf{x}^2$, $p^2 := (E/c)^2 - \mathbf{p}^2$ use the Minkowski signature. $\mathbf{p}\cdot\mathbf{x}$ is the dot product and $\mathbf{x}^2 := \mathbf{x}\cdot\mathbf{x}$ is the squared Euclidean length. The units of spacetime coordinates are length, and wavenumbers p have the units of inverse length. Mass m is in natural units and a relevant length associated with a mass m is the reduced Compton wavelength λ_c from (14). The Fourier transforms of generalized functions are defined [19] to satisfy Parseval's equality

$$\tilde{T}(\tilde{\psi}) := T(\psi). \tag{17}$$

As a consequence and when applicable, the Fourier transforms of generalized functions are

$$\tilde{T}(p) = \int \frac{dx}{(2\pi)^2}\, e^{ipx}T(x)$$

with the sign reversal of the exponent in the exponential function relative to (16). The Fourier transform is invertible [48].

$$\psi(x) = \int \frac{dp}{(2\pi)^2}\, e^{ipx}\tilde{\psi}(p).$$

Among the properties of Fourier transforms are the Fourier transform pairs

$$\begin{aligned} \psi(\Lambda(x-a)) &\leftrightarrow e^{-ipa}\tilde{\psi}(\Lambda p) \\ \frac{d\psi(x)}{dx} &\leftrightarrow igp\tilde{\psi}(p) \\ e^{-\alpha\|x\|^2} &\leftrightarrow \frac{1}{4\alpha^2}e^{-\|p\|^2/(4\alpha)} \end{aligned} \tag{18}$$

with $\|x\|^2$ the Euclidean (sum of squares) length squared of the Lorentz four-vector x, for a Lorentz transformation Λ, spacetime translation a, and complex α with $\Re e(\alpha) > 0$. g is the Minkowski signature,

$$gp = (p_0, -\mathbf{p}). \tag{19}$$

The third property in (18) suggests that functions $\psi(x)$ with broad spacetime support have Fourier transforms with concentrated energy-momentum support, and vice versa. Indeed, in general, the standard errors of $|\psi(x)|^2$ in spacetime and of $|\tilde{\psi}(p)|^2$ in energy-momenta considered as probability distributions satisfy $\sigma_x\sigma_p \geq \frac{1}{2}$ in each dimension. $\sigma_x\sigma_p \geq \frac{1}{2}$ illustrates the Heisenberg uncertainty principle discussed in appendix 6.7. More precise knowledge of the location of a body implies degraded knowledge of the time rate of change of the location, and this effect is more pronounced for low mass bodies than for heavy bodies due to $\hbar\mathbf{p} = \mathbf{P} \approx m\mathbf{v}$ at nonrelativistic velocities $\mathbf{v}$. Then

$$\sigma_x\sigma_v \geq \frac{\hbar}{2m}$$

in each of the three dimensions.

3.1.2 The scalar product

The Hilbert spaces of interest have a scalar product $\langle \underline{f}|\underline{g}\rangle \in \mathbb{C}$ for every pair of elements $|\underline{f}\rangle, |\underline{g}\rangle \in \mathbf{H}_{\mathcal{P}}$ described by function sequences $\underline{f}, \underline{g}$. The scalar product provides the norm,

$$\|\underline{f}\| := \sqrt{\langle \underline{f}|\underline{f}\rangle}. \tag{20}$$

To achieve Poincaré invariance of likelihoods and limit the support of states to nonnegative energies, the degenerate scalar product uses generalized functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \in \mathcal{S}'(\mathbb{R}^{4n})$ [20]. This degenerate scalar product is

$$\begin{aligned}
\langle \underline{f}|\underline{g}\rangle &:= \mathcal{W}(\underline{f}^*, \underline{g}) \\
&:= \sum_{n,m} \sum_{(\kappa)_{n+m}} \mathcal{W}_{n,m}(f^*_{n,(\kappa)_n}\, g_{m,(\kappa)_{n+1,n+m}})_{(\kappa)_{n+m}} \\
&= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\ (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\
&\qquad\qquad \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}\, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}
\end{aligned} \tag{21}$$

with formal summation notation for generalized functions [19] in the last line. Each spacetime Lorentz vector x_k is summed over $\mathbb{R}^4$ and each $\kappa_j \in \mathbb{N}$ is summed from 1 to N_c, the number of field components. The indices n and m are summed over the nonnegative integers, $\{0, \infty\}$. The VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ generalize the Wightman functions [9, 10, 56, 62]. Multiple arguments are denoted

$$(x)_{j,k} := x_j, x_{j+1}, \ldots x_k \in \mathbb{R}^{4(k-j+1)}$$

in the ascending case, $(x)_{j,k} := x_j, x_{j-1}, \ldots x_k$ otherwise and $(x)_n := (x)_{1,n}$. The $*$-dual $\underline{f}^*$ of a function sequence $\underline{f}$ uses complex conjugation, argument transpositions, and the nonsingular $N_c \times N_c$ linear transformation D from (8). The $*$-dual functions are

$$f^*_n((x)_n)_{(\kappa)_n} := (D^T\cdot)_n \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}$$

using the matrix notation (9) and then the Fourier transform[k] of $f^*_n((x)_n)_{(\kappa)_n}$ is (8). D, designated here as Dirac conjugation, is determined by representation of the Lorentz subgroup and satisfies

$$\overline{D}D = \mathbb{I}_{N_c} \tag{22}$$

with $\mathbb{I}_{N_c}$ the $N_c \times N_c$ identity and as a consequence, the $*$-dual satisfies

$$\begin{aligned} \underline{f}^{**} &= \underline{f} \\ (\underline{g} \times \underline{f})^* &= \underline{f}^* \times \underline{g}^*. \end{aligned}$$

The linear space of function sequences $\underline{\mathcal{P}}$ becomes an algebra with the $\times$-product (4). The $*$-dual is an involution of the algebra of function sequences when it is an automorphism. The $*$-dual maps $\underline{\mathcal{S}} \mapsto \underline{\mathcal{S}}$ but, the $*$-dual is not an automorphism for the $\underline{\mathcal{P}}$ selected in the constructions: $\underline{\mathcal{P}}^* \neq \underline{\mathcal{P}}$ due to a nonnegativity constraint on the energy support of the elements of $\mathcal{P}(\mathbb{R}^{4n})$. From the representation of the $*$-dual (8), $\underline{\mathcal{P}} \cup \underline{\mathcal{P}}^* \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{c\Omega\}$ with $c \in \mathbb{R}$ and Ω the vacuum. Discussed in sections 2 and 3.7, the elements of $\mathcal{P}(\mathbb{R}^{4n})$ have zeros on the negative energy mass shells and the $*$-dual maps these zeros to positive energy mass shells.

The notation $\langle f_n | g_m \rangle$ is used for a scalar product when the constituent functions $f_j = 0$ for $j \neq n$ and $g_k = 0$ when $k \neq m$ in the function sequences $\underline{f}$ and $\underline{g}$.

$$\begin{aligned} \langle f_n | g_m \rangle \;:=\; \sum_{(\kappa)_{n+m}} \int d(x)_{n+m}\, (D\cdot)_n \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ \times \overline{f_n}(x_n, \ldots x_1)_{\kappa_n \ldots \kappa_1}\, g_m(x_{n+1}, \ldots x_{n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned} \tag{23}$$

The sequence of generalized functions is denoted

$$\underline{\mathcal{W}} := (1, \mathcal{W}_{1,0}, \mathcal{W}_{0,1}, \ldots, \mathcal{W}_{n,0}, \mathcal{W}_{n-1,1} \ldots, \mathcal{W}_{0,n}, \ldots). \tag{24}$$

The arguments of $\mathcal{W}_{k,n-k}$ are $(x)_n, (\kappa)_n$ and similarly to (15), there are $(N_c)^n$ functions in the subsequence $\mathcal{W}_{k,n-k}$ distinguished by n. Each $\kappa_j \in \{1, N_c\}$ and $j \in \{1, n\}$. The n arguments of $\mathcal{W}_{k,n-k}$ are designated as k $*$-dual function and $n-k$ function arguments.

The scalar product of $\mathbf{H}_{\mathcal{P}}$ results from the isometry

$$\langle \underline{f} | \underline{g} \rangle = \mathcal{W}(\underline{f}^*, \underline{g})$$

that associates elements of the Hilbert space with equivalence classes of function sequences [12]. Then the norm (20) is

$$\|\underline{f}\| = (\mathcal{W}(\underline{f}^*, \underline{f}))^{1/2}.$$

[k] $\int dx\, e^{-ipx}\overline{\psi(x)} = \overline{\int dx\, e^{ipx}\psi(x)} = \overline{\tilde{\psi}(-p)}$.

3.1.3 The quantum field and VEV

The quantum field derives from the expression (6) for scalar product. This definition determines the VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ in the scalar product (21) from the VEV of the fields (6).

The quantum field is multiplication (4) in the algebra of function sequences. Hans-Jürgen Borchers [10] definition is

$$\Phi(\underline{f})\,\underline{g} := \underline{f} \times \underline{g} \tag{25}$$

with $\underline{f}, \underline{g} \in \mathbf{H}_{\mathcal{P}}$ but in this note, the discussion of $\Phi(\underline{f})$ is limited to single argument functions f_1 from sequences $\underline{f}$. Then, the quantum field consists of N_c field components $\Phi(x)_\kappa$

$$\begin{aligned}\Phi(\underline{f}) &:= \sum_{\kappa=1}^{N_c} \Phi(f_\kappa)_\kappa \\ &= \sum_{\kappa=1}^{N_c} \int dx\ \Phi(x)_\kappa f_1(x)_\kappa.\end{aligned}$$

with field component $\Phi(x)_\kappa$ associated with mass m_κ. Comparison of the scalar products (6) and (21) identifies

$$(D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} := \langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1} \Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n} \Omega \rangle \tag{26}$$

in the matrix notation (9). The definition (25) of quantum field produces (2) with states described by state describing function sequences (15).

If the multiplication (4) preserves Hilbert space norm-equivalence classes of function sequences, then the definition of field as multiplication of function sequences (25) elevates to Hilbert space operators. In Wightman's development [10, 56, 62], the basis function spaces are the $*$-algebra of tempered functions $\underline{\mathcal{S}}$ (totality W.b in section 3.2). Then, $\underline{g}^* \times \underline{h} \in \underline{\mathcal{S}}$ for $\underline{g}, \underline{h} \in \underline{\mathcal{S}}$ and if the scalar product satisfies formal Hermiticity ($\mathcal{W}_{k,n-k} = W_n$ independently of k, W.a in section 3.2), then the Cauchy-Schwarz-Bunyakovsky inequality demonstrates that the field (25) preserves equivalence classes. But, neither a $*$-involutive algebra of function sequences nor formal Hermiticity are necessary to realize relativistic quantum physics. The $*$-dual (8) is not an involution of the algebra $\underline{\mathcal{P}}$ used in this development, section 3.7 and [33, 37], and if the constructions are based on the $*$-algebra $\underline{S}$, then the constructed physically nontrivial VEV are not formally Hermitian. Nevertheless, the constructed quantum fields are unbounded Hilbert space operators, section 3.8.

There are no eigenfunctions of the quantum field (25). An eigenstate would be labeled by a sequence (15) of generalized functions $\underline{e}$ such that

$$\Phi(\underline{f})\,\underline{e} = \lambda\,\underline{e}.$$

For the case $N_c = 1$ and no vacuum polarization,

$$\begin{aligned}\Phi(\underline{f})(e_0, e_1, e_2, e_3 \ldots) &= (0, f, 0, \ldots) \times (e_0, e_1, e_2, e_3 \ldots) \\ &= (0, fe_0, fe_1, fe_2 \ldots) \\ &= (\lambda e_0, \lambda e_1, \lambda e_2, \lambda e_3 \ldots)\end{aligned}$$

from (4) and (25). If $\underline{e}$ describes an eigenstate of $\Phi(\underline{f})$ with a finite eigenvalue λ, then $\lambda\underline{e} - \underline{f} \times \underline{e}$ is a sequence in the null space of $\mathbf{H}_{\mathcal{P}}$. Selection of the sequence of zeroes to represent $\lambda\underline{e} - \underline{f} \times \underline{e}$ results in the recursive $\lambda e_{n+1} = fe_n$. Then $\lambda e_0 = 0$ with $\lambda \neq 0$ provides that $\underline{e} = 0$.

3.1.4 Likelihoods: Born's rule

A separable Hilbert space has a denumerable basis of orthonormal elements designated $|\underline{e}_\nu\rangle$ here [3, 40]. These elements provide a resolution of the identity operator in $\mathbf{H}_{\mathcal{P}}$,

$$\mathbb{I} = \sum_\nu Q_\nu$$

with projection operators

$$Q_\nu := |\underline{e}_\nu\rangle\langle\underline{e}_\nu|$$

in bra-ket notation. Then the expansion of any state in this selected basis is

$$|\underline{f}\rangle = \sum_\nu \langle\underline{e}_\nu|\underline{f}\rangle \; |\underline{e}_\nu\rangle.$$

The projection operators Q_ν are the elementary propositions [6, 60] of measurement and for normalized states $\|\underline{f}\| = 1$, the squared magnitudes $|\langle\underline{e}_\nu|\underline{f}\rangle|^2$ of the coefficients are the likelihoods that proposition ν is answered affirmatively. The elementary propositions are the queries "will the state described by $\underline{f}$ be perceived as the state described by $\underline{e}_\nu$." This identification of likelihoods is *Born's rule*.

Born's rule provides that the likelihood of observing the state described by $|g\rangle$ after interaction with a state initially described by $|\psi\rangle$ is the squared magnitude of the scalar product,

$$\text{likelihood} := |\langle g|\psi\rangle|^2 \leq 1$$

for normalized state descriptions, $\|g\| = \|\psi\| = 1$. Born's rule requires no additional assumptions for the forms or properties of operators to evaluate likelihoods: a scalar product and the description (15) of states $|\underline{f}\rangle$ are inherent to the Hilbert space.

3.2 Axioms for the VEV of relativistic fields

Early study of mathematical structures that might realize relativistic quantum physics includes studies by John von Neumann, Rudolf Haag, Res Jost, Arthur Wightman, Léon van Hove, Nikolay Bogolubov, Hans-Jürgen Borchers, Huzihiro Araki and Raphael Høegh-Krohn. Early studies did not identify structures that demonstrably realize nontrivial relativistic quantum physics [7, 8, 9]. Methods include the proposal of axioms to characterize general properties of relativistic quantum physics. Axioms are used to define concepts, derive general results, and assess the consistency of the axioms with additional assumptions. Significantly, the only realizations discovered for the established prospective axioms are physically trivial. Established axioms for relativistic quantum physics include the Wightman functional analytic axioms [9, 10, 56, 62], the Gårding-Wightman axioms for field operators [9, 56, 63], and the Haag-Kastler (Araki-Haag-Kastler) algebraic axioms for bounded, local Hermitian operators [9, 66, 67]. These axioms all consider Hilbert space realizations in addition to Fock space but the realizations that have been constructed do not exhibit interaction. The realizations of relativistic quantum physics in section 3.4 and [31, 33, 37] satisfy a revision to the Wightman axioms that preserves the physical characteristics and relaxes technical properties motivated by compliance with the canonical formalism. The realizations also satisfy the Haag-Kastler axioms.

From Born's rule, the scalar products of function sequences determine state transition likelihoods and these likelihoods exhibit physical properties. The scalar product (21) is determined by the VEV (26). Seven prospective axioms adapted from the Wightman functional analytic axioms [9, 10, 56, 62] and RQFT [61] characterize the constructions of relativistic quantum physics.

A.1) *Regularity:* the vacuum expectation values (VEV) of quantum fields are generalized functions dual to the Schwartz tempered functions $\underline{\mathcal{S}}$. The VEV satisfy axioms A.2-7 in a subspace $\underline{\mathcal{P}} \subseteq \underline{\mathcal{S}}$.

A.2) *Positive definiteness:* the state describing functions are elements of a Hilbert space $\mathbf{H}_{\mathcal{P}}$. VEV define (21) a degenerate scalar product $\mathcal{W}(\underline{g}^*, \underline{f})$, $\mathcal{W}(\underline{f}^*, \underline{f}) \geq 0$, for sequences of state describing functions $\underline{g}, \underline{f} \in \underline{\mathcal{P}}$. The $*$-dual sequence (8) of every function sequence $\underline{f} \in \underline{\mathcal{P}}$ is within $\underline{\mathcal{S}}$.

A.3) *Relativistic invariance:* transition likelihoods are the same for all inertial observers. The degenerate scalar product (21) is invariant to proper orthochronous Poincaré transformations of the function sequences.

A.4) *Spectral support:* energy-momenta lie within the closed forward (nonnegative energy) cone.

A.5) *Local commutativity:* field values are independent if not causally related and similarly described particles are indistinguishable. The magnitudes of scalar products are invariant with interchange of arguments of function sequences $\underline{f}$ if supports are space-like separated.

A.6) *Cluster decomposition:* transition amplitudes for non-entangled, distantly space-like separated state descriptions are independent. The degenerate scalar product for functions supported on distantly space-like separated volumes factors,

$$\mathcal{W}(\underline{g_1}^* \times \underline{g_2}^*, \underline{f_1} \times \underline{f_2}) \mapsto \mathcal{W}(\underline{g_1}^*, \underline{f_1})\mathcal{W}(\underline{g_2}^*, \underline{f_2}), \tag{27}$$

as the supports of $\underline{g_1}, \underline{f_1}$ become arbitrarily distantly space-like separated from the supports of $\underline{g_2}, \underline{f_2}$.

A.7) *Elemental stability:* Without vacuum polarization, the vacuum and one particle states are orthogonal to multiple argument states. $\langle \underline{g}|\Omega\rangle = 0$ and $\langle \underline{g}|\Phi(\underline{f})\Omega\rangle = 0$ if $\underline{g} \in \underline{\mathcal{P}}$ with $g_0 = g_1(x)_\kappa = 0$, $\kappa \in \{1, N_c\}$.

The notation is developed in section 3.1 and [10]. Axioms A.2-6 are designated here as the *physical conditions* [31]. Regularity and positive definiteness imply that states are realized as elements of a rigged Hilbert space and satisfaction of the physical conditions applies in the Hilbert space $\mathbf{H}_\mathcal{P}$ based on function sequences $\underline{\mathcal{P}}$.

Given generalized functions $\mathcal{U}_{n,m}$ that satisfy A.1-6, generalized functions $\mathcal{W}_{n,m}$ that also satisfy A.7 follow from

$$\mathcal{W}(\underline{f}^*, \underline{f}) := |f_0|^2 + \mathcal{W}_{1,1}(f_1^*, f_1) + \mathcal{U}((\underline{f} - {}^S\underline{f})^*, \underline{f} - {}^S\underline{f}) \tag{28}$$

with the shortened sequence ${}^S\underline{f} := (f_0, f_1(x)_\kappa, 0, 0, \ldots)$ a projection of $\underline{f}$. $\mathcal{W}_{1,1}$ is a free field two-point function. In (28), the vacuum polarizations are set to zero without loss of generality: finite vacuum polarizations are implemented in section 3.4.5. The decoupling of the contributions from the $n \geq 4$ argument VEV in $\underline{\mathcal{U}}$ from the two-point function of $\underline{\mathcal{W}}$ illustrates that with the revised axioms, a two-point function equal to the free field two-point function no longer implies that the quantum field is a free field. Without densely defined Hermitian field operators, that is, without VEV that satisfy the additional constraints of formal Hermiticity W.a and totality W.b introduced below, the Jost lemma [9, 56] does not apply to the constructions. If the field in (5) is Hermitian, densely defined and satisfies A.1-5, then VEV contributions such as (10) and (28) are precluded.

Using Born's rule, all inertial observers perceive the same likelihood of events if the scalar products are invariant to Poincaré transformations.

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \langle \underline{g}|\underline{f}\rangle, \tag{29}$$

for Poincaré transformations

$$\begin{aligned} (a,\Lambda)f_n((x)_n)_{(\kappa)_n} &:= (S(A)^T\cdot)_n f_n((\Lambda^{-1}(x-a))_n)_{(\kappa)_n} \\ (a,\Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} &= \prod_{k=1}^{n} e^{-ip_k a}\,(S(A)^T\cdot)_n \tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n} \end{aligned} \tag{30}$$

from the Fourier transform (16) with properties (18), with the matrix notation (9) and with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz group. $A \in \mathrm{SL}(2,\mathbb{C})$, Λ is the proper orthochronous Lorentz transformation determined by A, and $a \in \mathbb{R}^4$ is a translation in spacetime. The 4×4 Λ has components

$$\Lambda_{jk} = \frac{1}{2}\mathrm{Trace}(\sigma_j A \sigma_k A^\dagger) \tag{31}$$

with σ_j designating one of the four Pauli spin matrices [9]. $\Lambda^T g \Lambda = g$ with g the Minkowski signature matrix (19). In a nonrelativistic development, time is a universal parameter and many Hamiltonians are compatible with the nonrelativistic scalar product [52] while in relativistic physics, time translations of states, and hence, the Hamiltonian, must comply with Poincaré invariance of likelihoods.

With the Fourier transform (16) and Poincaré transformations (30), time translations by λ are

$$\tilde{f}_n((p)_n)_{(\kappa)_n} \to \prod_{j=1}^{n} e^{-ip_{j0}\lambda}\, \tilde{f}_n((p)_n)_{(\kappa)_n}$$

in the n-argument subspace of the momentum domain. Time translation advances the argument of the field positively in time, and as a consequence, the support of functions translate oppositely in time.

$$\int dx\ \Phi(x) f(x_0 - \lambda, \mathbf{x}) = \int dx\ \Phi(x_0 + \lambda, \mathbf{x}) f(x).$$

Poincaré invariance provides that temporal translation is unitarily implemented,

$$U(\lambda) f((x)_n)_{(\kappa)_n} := f((x_0 - \lambda, \mathbf{x})_n)_{(\kappa)_n}. \tag{32}$$

From the Fourier transform (16) with properties (18),

$$e^{-i\mathbf{p}\cdot\mathbf{y}} \tilde{f}(p_0, \mathbf{p} - \mathbf{q}) \tag{33}$$

is the inverse Fourier transform of

$$e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} f(x_0, \mathbf{x} - \mathbf{y}). \tag{34}$$

The support for the function in (33) is translated from $f(x)$ in space by $\mathbf{y} \in \mathbb{R}^3$ and in momentum by $\mathbf{q} \in \mathbb{R}^3$.

A.4 follows from the observation that energy-momenta Lorentz vectors lie within the closed forward cone

$$\overline{V}^+ := \{p :\ p^2 \geq 0 \text{ and } p_0 \geq 0\}.$$

If $|p\rangle$ is a generalized eigenstate of momentum with eigenvalue p, then for any state describing function g_k of k arguments,

$$\langle g_k | p \rangle = 0$$

if $p \neq \overline{V}^+$. As a consequence, Fourier transforms of the generalized functions

$$\tilde{T}_n((p)_n)_{(\kappa)_n} := \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} \widetilde{g_k^*}((p)_k)_{(\kappa)_k} \tilde{f}_{n-k}((p)_{k+1,n})_{(\kappa)_{k+1,n}} \tag{35}$$

are limited to $\mathcal{E}_n^+$ [9, 10, 56, 62]. $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are the VEV functions (26), $g_k \in \mathcal{P}(\mathbb{R}^{4k})$, $f_{n-k} \in \mathcal{P}(\mathbb{R}^{4n-4k})$ and

$$\mathcal{E}_n^+ := \{(p)_n : \ p_n \in \overline{V}^+, \ p_{n-1} + p_n \in \overline{V}^+, \ldots, \ p_2 + \ldots p_n \in \overline{V}^+, \ p_1 + p_2 + \ldots p_n = 0\}. \tag{36}$$

The support limitation follows from spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [40], chapters 7-10 [24]). The unitary spacetime translation operator in $\mathbf{H}_\mathcal{P}$ is

$$U(a) = \int dE(p) \ e^{-ipa}$$

with

$$dE(p) \sim dp \ |p\rangle\langle p|$$

a resolution of the identity in $\mathbf{H}_\mathcal{P}$. Every $p \in \overline{V}^+$. Unitarity of $U(a)$ follows from Poincaré invariance A.3 of the scalar product and Hermiticity of the densely defined generators of translations, the energy-momentum, appendix 6.3, follows from Stone's theorem. Translations of the fields are

$$U(a)\Phi(x_j)_{\kappa_j}U(a)^{-1} = \Phi(x_j + a)_{\kappa_j},$$

and the vacuum is translation invariant.

$$U(a)\Omega = \Omega.$$

Then, translation of arguments ℓ though n is expressed

$$\begin{aligned} &\langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell U(a)^{-1} \ldots U(a) \Phi_n U(a)^{-1} \Omega \rangle \\ &\qquad = \ \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell \ldots \Phi_n \Omega \rangle \end{aligned}$$

with an abbreviated notation for fields $\Phi_j := \Phi(x_j)_{\kappa_j}$. The scalar product (6) with functions g_{n-k} of translated support for arguments ℓ through n provides

$$\begin{aligned} F(q) \ &:= \int da \ e^{iqa} \int d(x)_n \ \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} U(a) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} g_{n-k} \\ &= \int da \int d(x)_n \int e^{iqa - ipa} \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} dE(p) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} f_{n-k} \\ &= (2\pi)^4 \int d(x)_n \int \delta(q-p) \ \langle \Phi_k \ldots \Phi_1 \Omega | \Phi_{k+1} \ldots \Phi_{\ell-1} dE(p) \Phi_\ell \ldots \Phi_n \Omega \rangle \ \overline{f_k} g_{n-k} \\ &= 0 \end{aligned}$$

if $q \notin \overline{V}^+$ for each choice of n, k, ℓ and $(\kappa)_n$. From the equivalent expression (21) for scalar product, Parseval's equality (17) and the properties of Fourier transforms (18),

$$\begin{aligned} F(q) &:= \int da\, e^{iqa} \int d(x)_n\, \mathcal{W}_{k,n-k}((x)_{\ell-1}, (x+a)_{\ell,n})_{(\kappa)_n}\, f_k^* g_{n-k} \\ &= \int da \int d(p)_n\, e^{iqa - i\sum_{j=\ell}^n p_j a}\, \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}\, \widetilde{f_k^*} \tilde{g}_{n-k} \\ &= (2\pi)^4 \int d(p)_n\, \delta(q - \textstyle\sum_{j=\ell}^n p_j)\, \widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}\, \widetilde{f_k^*} \tilde{g}_{n-k}. \end{aligned}$$

Then, $F(q) = 0$ unless

$$\sum_{j=\ell}^n p_j \in \overline{V}^+$$

implies that the support of (35) includes only p_j from $\mathcal{E}_n^+$ (36). Development follows similarly for $\ell \leq k$ from the sesquilinearity of the scalar product (6). As a consequence, to satisfy the observation that all energy-momentum lie in the closed forward cone, the support of (35) includes only p_j from $\mathcal{E}_n^+$.

In Wightman's original axioms, the support of the functions $\tilde{f}_k, \tilde{g}_{n-k}$ were not constrained and as a consequence, the support of the VEV functions $\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n}$ was limited to $\mathcal{E}_n^+$. The supports of the constructed VEV are limited to mass shells and the supports of elements of $\mathcal{P}(\mathbb{R}^{4n})$ are limited to positive energies. Together, the joint support of VEV functions and functions in $\underline{\mathcal{P}}$ eliminates the negative energy mass shells and satisfies A.4 [33]. This development enables joint satisfaction of locality, positive energy support and Poincaré covariance, an unmet and apparently unattainable task within RQFT developments when interaction is manifest [4, 9, 31, 41]. Satisfaction of the physical conditions in a more constrained function space (or equivalently, with VEV that do not satisfy formal Hermiticity) results in the constructed physically nontrivial, locally commutative, positive energy, Poincaré covariant tempered distribution-valued quantum fields.

Local commutativity is that linear combinations of VEV with transpositions of arguments conditionally vanish.

$$\langle \underline{f} | \Phi_1 \dots (\Phi_k \Phi_{k+1} \pm \Phi_{k+1} \Phi_k) \dots \Phi_n \Omega \rangle = 0$$

if the points x_k and x_{k+1} are space-like, $(x_k - x_{k+1})^2 < 0$, $1 \leq k < n$, and the notation is abbreviated, $\Phi_k := \Phi(x_k)_{\kappa_k}$. Space-like separations specify that the values of the quantum field are not causally related. The sign is determined by particle statistics to satisfy normal commutation relations [9]. Verification of local commutativity uses functions of bounded support with space-like separated support, for example, tempered test functions $\underline{\mathcal{S}}$. The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ lack functions of bounded support and the constructed VEV exhibit unconditional signed symmetry for functions in $\underline{\mathcal{P}}$, section 3.4 and [31, 33]. For the realizations constructed in these notes, A.5

can be tightened to unconditional signed symmetry of VEV but to include free field developments with the basis function spaces extended to $\underline{\mathcal{S}}$, only conditional local commutativity is required in A.5 [31]. Inclusion of parastatistics [9] would extend this development.

Condition A.6 is another manifestation of causality: great spatial separation implies independence of the local observables of nonentangled subsystems. A.6 is stronger than the condition used in [31, 33, 37] that was also designated as cluster decomposition. Earlier constructions in [31, 33, 37] satisfy formal Hermiticity described below. However, in section 3.4 it is illustrated that formal Hermiticity conflicts with cluster decomposition A.6 in constructions with nonzero spin and in some scalar field examples. Both conditions, A.6 or the cluster decomposition condition used in [31, 33, 37], imply that the vacuum is the sole translation-invariant state. In common with Wightman's development, formal Hermiticity was maintained in [37] and resulted in truncated functions that are not connected. In the earlier development that satisfied formal Hermiticity, the strong cluster decomposition condition A.6 is not satisfied. The constructions in section 3.4 abandon formal Hermiticity to establish the causal cluster decomposition property A.6. Cluster decomposition is a nonlinear condition and relates the description of interaction across orders of the VEV.

Condition A.6 implies that: the vacuum $|\Omega\rangle$ is the only translationally invariant state [33, 56]; that states with sufficiently isolated and space-like separated support are described by free particles, section 3.6; that the quantum field (25) elevates to Hilbert space operators, section 3.8; and the essential independence of the local observables of non-entangled, spatially distant bodies, appendices 6.2.7 and 6.2.8.

For the statement of A.7 and in the example constructions, mean values of field components, vacuum polarizations, $\langle\Omega|\Phi(x)_\kappa\Omega\rangle$, are asserted to vanish without loss of generality. Mean values of the boson field components are independently specified constants. Nonzero mean values are introduced in section 3.4.5.

Condition A.7 provides an interpretation of states as free particles in appropriate instances and results in satisfaction of regularity A.1. Condition A.7 together with A.6 and the selected two-point functions provide that isolated and localized volumes of support of the state describing functions are perceived as classical particles. This associates the quantum field with elementary particles. Sufficiently isolated concentrations of support propagate as nearly free particles. The masses of the elementary particles are the m_κ associated with each field component $\Phi(x)_\kappa$ in the construction in section 3.4. A.7 provides satisfaction of positive definiteness A.2 together with regularity A.1 for the constructions. However, satisfaction of A.7 is not necessary for regularity. Regularity and nonnegativity require that $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ for $n \geq 2$ in the constructions, section 3.5.5. Satisfaction of formal Hermiticity precludes setting $\mathcal{W}_{0,n} = \mathcal{W}_{1,n} = 0$ for $n \geq 2$ but *kinematic constraints* can result in $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ if $f_1 \in \mathbf{H}_\mathcal{P}(\mathbb{R}^4)$ and $f_n \in \mathbf{H}_\mathcal{P}(\mathbb{R}^{4n})$ for nonzero $\mathcal{W}_{0,n}$ and $\mathcal{W}_{1,n}$. Kinematic constraints include conservation of energy-momentum and spin. For the realizations with only a single mass, $\mathcal{W}_{0,n}(f_0^*, f_n) = \mathcal{W}_{1,n}(f_1^*, f_n) = 0$ if $n \geq 2$ due to conservation of energy-momentum: neither the vacuum nor a single particle can create a cascade of particles with each particle of the same

mass as the decaying particle.

Confined particles, those that do not appear with their support isolated from the support of other arguments, have $\mathcal{W}_{1,1} = 0$ for the κ_j included in their description. A $|f_0|^2$ term ($\mathcal{W}_{0,0} = 1$) is required in any construction that includes a vacuum state. Elemental stability A.7 expresses that if a state is prepared with only one elementary particle, or together with A.6 that if one elementary particle is greatly isolated from the support of every other body, then the elementary particle is stable until it encounters another body. Whether or which elementary particles are confined is not determined by the axiom and a discussion of confinement awaits better understanding of bound states within the constructions. Confined species extends this development.

Wightman's original axioms [9, 10, 56, 62] are distinguished here by two additional conditions.[1] Wightman's axioms support the canonical formalism's conjectured correspondence that quantizes classical fields to Hermitian quantum field operators, Wightman's prospective axioms include assumptions that imply the fields in the VEV (5) are Hermitian Hilbert space field operators:

W.a) *Formal Hermiticity:* the generalized functions that define the degenerate scalar product (21) satisfy

$$\mathcal{W}_{k,n-k}((x)_n))_{(\kappa)_n} = W_n((x)_n))_{(\kappa)_n}$$

independently of k. In this case, the degenerate scalar product is

$$\mathcal{W}(\underline{g}^*, \underline{f}) = \underline{W}(\underline{g}^* \times \underline{f})$$

for a Wightman functional $\underline{W}$ and the product (4).

W.b) *Totality:* $\underline{\mathcal{P}} = \underline{\mathcal{S}}$. Satisfaction of the axioms applies for all sequences of Schwartz tempered functions.

Together with A.1-2, satisfaction of W.a and W.b implies that the field is realized as Hermitian Hilbert space operators. Condition W.a results in the simplified form (7) for the scalar product. Sequences of tempered functions $\underline{\mathcal{S}}$ are a $*$-algebra for the product (4) and involution (8). However, satisfaction of both conditions W.a-b preclude the constructed nontrivial realizations of relativistic quantum physics. Physically trivial examples, free fields, Wick polynomials and generalized free fields, satisfy both W.a and W.b. Either, but not both W.a and W.b can be satisfied by the physically nontrivial constructions in section 3.4. In section 3.7.2, it is demonstrated that W.b can be satisfied if W.a is abandoned: with a non-Hermitian embellishment to the VEV presented in section 3.4, the constructions can be based on $\underline{\mathcal{S}}$. In [31], it is demonstrated that W.a can be satisfied for a physically nontrivial neutral scalar field if

[1] The conditions are not called out separately as axioms but formal Hermiticity and totality are assumed within a statement of the axioms [9, 10, 56, 62].

W.b is abandoned. The departure from satisfaction of W.a-b admits the physically nontrivial constructions.

The revised axioms for RQP result in substantial departures from established RQFT results: without densely defined Hermitian field operators, "no go" theorems such as "no simple two-point functions" from Jost-Schroer and Greenberg [9] and "no nontrivial, infinitely divisible descriptions" from Rinke [47] no longer apply; and there are no states of strictly bounded support described by the energy support constrained functions in $\underline{\mathcal{P}}$ [33, 45, 53] although $\underline{\mathcal{P}}$ includes states that are arbitrarily dominantly supported in bounded volumes, appendix 6.14. The RQFT demonstrations of PCT and spin-statistics theorems [9, 56] require review to include the physically nontrivial RQP constructions.

3.3 Relativistic free fields

3.3.1 Free field VEV

Free field VEV satisfy both the original Wightman axioms [9, 10, 56, 62] and the prospective axioms A.1-7.

Satisfaction of the Wightman axioms suffices to determine that the neutral scalar field two-point function $W_2(x_1, x_2)$ is a Källén-Lehmann form, a summation over a mass spectrum of the Pauli-Jordan function [54]. The two-point functions used in the constructions are extensions of the Pauli-Jordan function to non-zero spin and have Fourier transforms

$$\begin{aligned} \tilde{W}_2(p_1, p_2)_{\kappa_1\kappa_2} &= \delta(p_1 + p_2)\ \delta_2^+\ M(p_2)_{\kappa_1\kappa_2} \\ &= \delta(\mathbf{p}_1 + \mathbf{p}_2)\ \delta_1^-\delta_2^+\ 2\sqrt{\omega_1\omega_2}\ M(p_2)_{\kappa_1\kappa_2} \end{aligned} \tag{37}$$

with

$$\delta_k^\pm := \theta(\pm E_k)\delta(p_k^2 - \lambda_{ck}^{-2})$$

and λ_{ck} is the reduced Compton wavelength (14) for mass m_{κ_k}. The supports of $\delta_k^\pm$ are on the positive $(+)$ or negative $(-)$ energy mass shells. A mass m_κ is associated with each field component $\Phi(x)_\kappa$.

The free field VEV satisfy formal Hermiticity W.a and totality W.b. The properties of the free field VEV suffice to imply that the quantum field (25) is a densely defined Hermitian Hilbert space operator and indeed, the VEV generate algebraically from the Hermitian field operator

$$\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa + \Phi_o^-(f)_\kappa$$

with creation and annihilation components that have commutation relations

$$[\Phi_o^-(f_1)_{\kappa_1}, \Phi_o^+(f_2)_{\kappa_2}]_\pm = W_2(f_1\, f_2)_{\kappa_1\kappa_2}$$

and otherwise commute, and there is a cyclic vacuum state $|\Omega_o\rangle$ with

$$\langle\Omega_o| \ldots \Phi_o^-(f)_\kappa \Omega_o\rangle = 0$$

[23, 24, 52, 61]. The notation is that Φ_o^+ is this creation and Φ_o^- is the annihilation component. Whether the commutator or anti-commutator is used depends on the values of κ_1, κ_2 that determine the statistics type of the field component, boson or fermion, as specified below in (41). In this algebraic evaluation of VEV, each $f_j \in \mathcal{S}(\mathbb{R}^4)$ [31]. From (26), the VEV functions and VEV of the field operators are related

$$(D\cdot)_k \mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n} := \langle \Phi_o(x_k)_{\kappa_k} \dots \Phi_o(x_1)_{\kappa_1} \Omega | \Phi_o(x_{k+1})_{\kappa_{k+1}} \dots \Phi_o(x_n)_{\kappa_n} \Omega \rangle.$$

Contrast of the two representations for the scalar product (6) and (21) with the definition of adjoint operator provides and condition (22) provide that

$$\Phi_o(x_j)^*_{\kappa_j} = D\Phi_o(x_j)_{\kappa_j}$$

with D from (8) and in the matrix notation (9). Then

$$\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n} = \langle \Omega_o | \prod_{j=1}^{n} \Phi_o(x_j)_{\kappa_j} \Omega_o \rangle$$

using the adjoint field and condition (22). The result of the algebraic evaluations for the VEV of the free field is that

$$\mathcal{F}_{k,2n-k}((x)_n)_{(\kappa)_n} = \sum_{pairs} \sigma(S,(\kappa)_n) \prod_{j=1}^{n} W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}} \tag{38}$$

with the summation over all $(2n)!/(2^n n!)$ ways of pairing the indices $i_j, \ell_j \in \{1, 2n\}$ without regard to order and the indices are ordered $i_j < \ell_j$ within $W_2(x_{i_j}, x_{\ell_j})_{\kappa_{i_j}\kappa_{\ell_j}}$. The sign of each term $\sigma(S,(\kappa)_n) = \pm 1$ is determined by particle statistics from the types of the indices $(\kappa)_n$. The $\sigma(S,(\kappa)_n)$ are positive if all indices $(\kappa)_n$ are boson indices; fermions introduce sign changes from the anti-commutation of $\Phi_o(x_j)^{\pm}_{\kappa_j}$ that achieve the pairing of indices $S = \{(i_1, \ell_1), \dots (i_n, \ell+n)\}$. The $\mathcal{F}_{k,n-k}((x)_n)_{(\kappa)_n}$ with an odd number of arguments n are zero. $\mathcal{F}_{0,0} = 1$.

The array $M(p)_{\kappa_1\kappa_2}$ in the two-point function (37) is used to construct the n-point connected functions. The elements of the $N_c \times N_c$ array $M(p)_{\kappa_1\kappa_2}$ are multinomials in the energy-momentum components [9] and in matrix notation, the array satisfies

$$DM(p) = C^\dagger(p)C(p) \tag{39}$$

with D the Dirac conjugation matrix from (8) and $C^\dagger(p)$ is the Hermitian transpose of $C(p)$. The matrix nonnegativity [27] of $DM(p)$ provides the degenerate scalar product (21) [37]. The nonnegativity of the degenerate scalar product (21) with the free field VEV $\underline{\mathcal{F}}$ is demonstrated in appendix 6.9 for the basis function spaces $\underline{\mathcal{P}}$. Since $DM(p)$ is a nonnegative matrix, it is Hermitian and it follows from the condition (22) for D that

$$M(p)^\dagger = DM(p)D^T. \tag{40}$$

Local commutativity A.5 is satisfied if the $M(p)$ are (real orthogonal) permutation matrix similar to a direct sum decomposition into two components,

$$OM(p)O^T = \begin{pmatrix} M_1(p) & 0 \\ 0 & M_2(p) \end{pmatrix}, \tag{41}$$

that satisfy locality conditions

$$M_j(p)^T = \pm M_j(-p). \tag{42}$$

The + sign applies for the boson constituent $M_1(p)$ and the − sign applies for the fermion constituent $M_2(p)$. (41) assigns a type, boson or fermion, to the value of each index $\kappa_j \in \{1, N_c\}$ for each j. The block diagonal array $M(p)$ and consequently W_2 vanishes unless both indices κ_1, κ_2 are the same type, boson or fermion. Poincaré invariance of the scalar product is implied by two additional conditions on $M(p)$ and D.

$$\begin{aligned} S(A)M(p)S(A)^T &= M(\Lambda^{-1}p) \\ \overline{S(A)}D &= DS(A) \end{aligned} \tag{43}$$

with $S(A)$ an $N_c \times N_c$ realization of the proper orthochronous Lorentz subgroup (30). $A \in \mathrm{SL}(2,\mathbb{C})$. Poincaré invariance is demonstrated in section 3.5.3. Example $M(p)$, D and $S(A)$ that satisfy (22), (39), (42) and (43) are illustrated in section 3.3.2.

$\underline{\mathcal{F}}$ exhibits local commutativity. Condition (42) implies commutation or anti-commutation of free field components. The two-point function (37), the Fourier transform convention (16), and translation invariance of the VEV express the local commutativity condition for free fields as

$$\begin{aligned} \langle\Omega|\left(\Phi_o(x)_{\kappa_1}\Phi_o(0)_{\kappa_2} \mp \Phi_o(0)_{\kappa_2}\Phi_o(x)_{\kappa_1}\right)\Omega\rangle &= \int dp\ \delta^+(p)\left(e^{-ipx}M(p)_{\kappa_1\kappa_2} \mp e^{ipx}M(p)_{\kappa_2\kappa_1}\right) \\ &= \int dp\ e^{-ipx}\left(\delta^+(p)M(p)_{\kappa_1\kappa_2} \mp \delta^-(p)M(-p)_{\kappa_2\kappa_1}\right) \\ &= \int dp\ e^{-ipx}M(p)_{\kappa_1\kappa_2}\left(\delta^+(p) - \delta^-(p)\right) \\ &= M(i\tfrac{d}{dx})_{\kappa_1\kappa_2}\int dp\ e^{-ipx}\left(\delta^+(p) - \delta^-(p)\right) \end{aligned}$$

with $x := x_1 - x_2$ and $p := p_2$, after reflection of the summation variable $p \mapsto -p$ in the second term, substitution of (42), and the properties of the Fourier transform (18) applied to the elements of the array $M(p)$ that are multinomials in the components of p. If $x^2 < 0$, then there is a Lorentz transformation Λ with $\Lambda x = -x$ and as a consequence, the Pauli-Jordan commutator function vanishes for space-like x. It then follows that the commutator/anti-commutator of free field components generally vanishes for space-like x. This local commutativity is satisfied for functions from $\underline{\mathcal{S}}$. $\underline{\mathcal{S}}$ includes functions with bounded support. $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ but $\underline{\mathcal{P}}$ includes none of the functions with bounded support.

If $f(x) \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$, then $\Phi_o(f)_\kappa = \Phi_o^+(f)_\kappa$ and $\Phi_o(f^*)_\kappa = \Phi_o^-(f^*)_\kappa$. As a consequence, if $g_k, f_{n-k} \in \underline{\mathcal{P}}$, the only contributions $\mathcal{F}_{k,n-k}(g_k^* f_{n-k})$ to scalar products are from

$$(D\cdot)_k \mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_S \sigma(S, (\kappa)_{2k}) \prod_{j=1}^{k} W_2(x_j, x_{i_j})_{\kappa_j \kappa_{i_j}}. \tag{44}$$

The summation $\sum_S$ includes the $k!$ distinct pairings j, i_j with $j \in \{1, k\}$ and $i_j \in \{k+1, 2k\}$. The scalar product (6) and definition (38) provide that $\sigma(S_o, (\kappa)_{2k}) = 1$ for the pairing

$$S_o = \{k, k-1, \ldots 1, k+1, k+2, \ldots 2k\}.$$

For $\underline{f} \in \underline{\mathcal{P}} \subset \underline{\mathcal{S}}$, only $k!$ of the $(2k)!/(2^k k!)$ terms for $n = 2k$ in (38) contribute to the free field VEV: terms in evaluations of the scalar product (21) that include two-point function argument pairings with a second argument from $\{1, k\}$ or a first argument from $\{k+1, 2k\}$ do not contribute. For the two-point function (37) and scalar product (21), and because functions in $\mathbf{H}_{\mathcal{P}}$ lack support on the negative energy mass shells,

$$W_2(f^*\, g^*)_{\kappa_1\kappa_2} = W_2(f\, g)_{\kappa_1\kappa_2} = W_2(f\, g^*)_{\kappa_1\kappa_2} = 0$$

and only $W_2(f^*\, g)_{\kappa_1\kappa_2}$ contributes if $f, g \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^4)$. These properties of the free field VEV for function sequences $\underline{\mathcal{P}}$ provide that $\underline{\mathcal{F}}$ exhibits an unconditional signed symmetry, (48) in section 3.4.2, that satisfies local commutativity A.5.

The free field VEV (44) result in scattering amplitudes with particle number conserved, and incoming and outgoing momenta equal in pairs, that is, no exhibition of interaction. Constructed from the two-point function (37) and the basis function space $\underline{\mathcal{S}}$, the free fields $\Phi_o(\underline{f})$ (25) are Hermitian Hilbert space operators that realize the Wightman, Gårding-Wightman and Haag-Kastler axioms [9, 10, 56, 62]. Constructed from (37) and the basis function spaces $\underline{\mathcal{P}}$, the free fields $\Phi_o(\underline{f})$ are not Hermitian operators but an available extension of the basis function spaces to $\underline{\mathcal{S}}$ achieves Hermitian field operators for the free field [31]. Hermitian Hilbert space field operators appear to be peculiar to physically trivial realization. Physically trivial relativistic realizations include generalized free and Wick polynomial (monomial) fields [9, 56].

3.3.2 Example $M(p)$, $S(A)$ and D

Realizations of free fields are available in [7, 9, 22, 23, 24, 52, 61]. The constructed nontrivial VEV $\underline{\mathcal{W}}$ employ a Dirac conjugation D, a realization for a two-point function $M(p)$, and a realization of the Lorentz subgroup $S(A)$, $A \in \mathrm{SL}(2, \mathbb{C})$, from a free field. These $N_c \times N_c$ matrices $D, M(p), S(A)$ satisfy (22), (39), (42) and (43).

A neutral scalar field is realized by $M(p) = S(A) = D = 1$, the scalar realization of the Lorentz subgroup. A charged scalar field is realized by two component fields ($N_c = 2$) with

$S(A) := \mathbb{I}_2$, the 2×2 identity, and

$$\begin{aligned} D &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ M(p) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M(-p)^T. \end{aligned}$$

This field has a symmetry associated with a charge, $S_\phi M(p) S_\phi^T = M(p)$.

$$S_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$.

A massive vector boson field is realized with four component fields ($N_c = 4$). This spin-1 boson field uses a real 4×4 representation of the Lorentz subgroup.

$$\begin{aligned} D &:= \mathbb{I}_4 \\ S(A) &:= \Lambda \\ M(p)_{jk} &:= \frac{p_{(j)} p_{(k)}}{m^2} - g_{jk} \\ &= C^\dagger(p) C(p). \end{aligned}$$

$p := (p_{(0)}, p_{(1)}, p_{(2)}, p_{(3)})$ and parentheses were used to distinguish energy-momentum vector components from energy-momentum vectors p_j. Here, m is shorthand for the inverse of the Compton wavelength $mc/\hbar$. g is the 4×4 Minkowski signature matrix (19) and Λ is a Lorentz transformation determined by $A \in \mathrm{SL}(2, \mathbb{C})$ in (31). $M(p) = M(-p)^T$. In four dimensions with $p := (p_0, \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z)$,

$$C^\dagger(p) = \begin{pmatrix} \frac{\sqrt{p_0^2 - m^2}}{m} & 0 & 0 & 0 \\ \frac{p_0}{m} \frac{\mathbf{p}_x}{\sqrt{p_0^2 - m^2}} & \sqrt{\frac{\mathbf{p}_y^2 + \mathbf{p}_z^2}{p_0^2 - m^2}} & 0 & 0 \\ \frac{p_0}{m} \frac{\mathbf{p}_y}{\sqrt{p_0^2 - m^2}} & \frac{-\mathbf{p}_x \mathbf{p}_y}{\sqrt{(\mathbf{p}_y^2 + \mathbf{p}_z^2)(p_0^2 - m^2)}} & \frac{\mathbf{p}_z}{\sqrt{\mathbf{p}_y^2 + \mathbf{p}_z^2}} & 0 \\ \frac{p_0}{m} \frac{\mathbf{p}_z}{\sqrt{p_0^2 - m^2}} & \frac{-\mathbf{p}_x \mathbf{p}_z}{\sqrt{(\mathbf{p}_y^2 + \mathbf{p}_z^2)(p_0^2 - m^2)}} & \frac{-\mathbf{p}_y}{\sqrt{\mathbf{p}_y^2 + \mathbf{p}_z^2}} & 0 \end{pmatrix}.$$

A massive spinor fermion field is also realized by four component fields ($N_c = 4$). This

spin-1/2 fermion field uses a 4×4 complex representation of the Lorentz subgroup.

$$\begin{aligned} D &:= \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \\ S(A) &:= \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \\ M(p) &:= \begin{pmatrix} 0 & A_P \\ A_P^T & 0 \end{pmatrix} \\ A_P &= A_P^\dagger := \begin{pmatrix} p_0 + \mathbf{p}_z & \mathbf{p}_x + i\mathbf{p}_y \\ \mathbf{p}_x - i\mathbf{p}_y & p_0 - \mathbf{p}_z \end{pmatrix} \end{aligned}$$

with 2×2 A_P, $\mathbb{I}_2$ and $A \in \mathrm{SL}(2, \mathbb{C})$. $M(-p) = -M(p)^T$.

$$DM(p) = C^\dagger(p)C(p) = \begin{pmatrix} c^T(p) & 0 \\ 0 & c^\dagger(p) \end{pmatrix} \begin{pmatrix} \overline{c}(p) & 0 \\ 0 & c(p) \end{pmatrix}$$

is a nonnegative matrix for $p := (p_0, \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z)$ within the forward cone and

$$c(p) = \begin{pmatrix} \sqrt{p_0 + \mathbf{p}_z} & \frac{\mathbf{p}_x + i\mathbf{p}_y}{\sqrt{p_0 + \mathbf{p}_z}} \\ 0 & \frac{m}{\sqrt{p_0 + \mathbf{p}_z}} \end{pmatrix}.$$

This field also has a symmetry $S_\phi M(p) S_\phi^T = M(p)$ associated with a charge,

$$S_\phi := \begin{pmatrix} \phi & 0 \\ 0 & \overline{\phi} \end{pmatrix} \qquad \phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

and $DS_\phi = \overline{S}_\phi D$.

The mixed product property of the direct sum $(U \oplus V)(U' \oplus V') = (UU' \oplus VV')$ and the Kronecker product $(U \otimes V)(U' \otimes V') = (UU' \otimes VV')$ [28] provides that direct sums and Kronecker (tensor) products of $M(p), S(A), D$ that satisfy (22), (39), (42), (43) are also solutions. Real orthogonal similarity transforms $OM(p)O^T, OS(A)O^T, ODO^T$ preserve solutions. Compositions of these example representations of $M(p), S(A), D$ and their sub-representations provide a rich class of realizations of relativistic quantum physics.

3.4 Nontrivial relativistic fields

VEV that exhibit nontrivial relativistic physics are displayed in this section. The construction generalizes the scalar example (10). The constructed VEV are cluster expansions of signed

symmetric, connected VEV functions ${}^C\underline{\mathcal{W}}$. The VEV satisfy properties A.1-7 and have forms suggested by earlier studies [2, 39] and by low order terms from Feynman series [31, 35].

From the definition of scalar product (21) and field (25), the VEV are generalized functions (26) designated

$$\langle \Phi(x_k)_{\kappa_k} \dots \Phi(x_1)_{\kappa_1}\Omega | \Phi(x_{k+1})_{\kappa_{k+1}} \dots \Phi(x_n)_{\kappa_n}\Omega\rangle = (D\cdot)_k \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$$

with Dirac conjugation D introduced in the $*$-dual of function sequences (8) and using the matrix notation (9).

3.4.1 Connected functions

The VEV fnctions are cluster expansions of connected functions. Connected functions include the two-point functions (37) of free fields and four or more argument connected functions illustrated by (10).

A *connected function*, ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$, attenuates with great space-like separation of its arguments. A connected function does not significantly contribute to scalar products if the support of $f_n((x)_n) \in \mathcal{S}(\mathbb{R}^{4n})$ consists of two greatly space-like separated volumes. That is, ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ is connected if

$$\lim_{\rho\to\infty} {}^C\mathcal{W}_{k,n-k}(f_n(\rho))_{(\kappa)_n} = 0 \tag{45}$$

with

$$f_n(\rho) := f_n(x_1, \dots x_j, x_{j+1} - \rho a, \dots, x_n - \rho a).$$

The Lorentz vector a is space-like ($a^2 < 0$), $\rho \in \mathbb{R}$, and $1 \le j < n$. (45) applies for all $(N_c)^n$ components labeled by the κ_j. In the scalar product (21),

$$f_n((x)_n) = h_k^*(((x)_k) g_{n-k}((x)_{k+1,n}) \in \mathcal{S}(\mathbb{R}^{4n})$$

with $h_k \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4k})$ and $g_{n-k} \in \mathbf{H}_{\mathcal{P}}(\mathbb{R}^{4(n-k)})$.

The connected functions used in the constructions are designated

$$\begin{aligned} {}^C\mathcal{W}_{1,1}(x_1,x_2)_{\kappa_1\kappa_2} &:= W_2(x_1,x_2)_{\kappa_1\kappa_2} \\ {}^C\widetilde{\mathcal{W}}_{k,n-k}((p)_n)_{(\kappa)_n} &:= c_n \delta(p_1 + \dots p_n)\, Q_{k,n-k}((p)_n)_{(\kappa)_n} \end{aligned} \tag{46}$$

with $2 \le k \le n-2$, $n \ge 4$ and even, $n = 2\ell$ for $\ell \in \mathbb{N}$. W_2 is a free field two-point function (37) from section 3.3.1, and the elements of the Lorentz covariant array $Q_{k,2n-k}$ are products of delta functions supported on mass shells for finite masses m_{κ_j} and regular functions of the energy-momenta. $Q_{k,2n-k}$ is displayed in (55) of section 3.4.3. To satisfy Lorentz covariance A.3 for nonzero spins, odd orders of the VEV vanish.

$${}^C\mathcal{W}_{k,2\ell-1-k}((x)_{2\ell-1})_{(\kappa)_{2\ell-1}} := 0$$

for all k. To satisfy elemental stability A.7,

$$\mathcal{W}_{0,k} = \mathcal{W}_{k,0} = \mathcal{W}_{1,k} = \mathcal{W}_{k,1} = 0$$

for $k \geq 2$ and for all κ_j. A.7 follows from the cluster expansion, section 3.5.4, and setting

$${}^{C}\mathcal{W}_{0,k} = {}^{C}\mathcal{W}_{k,0} = {}^{C}\mathcal{W}_{1,k} = {}^{C}\mathcal{W}_{k,1} = 0$$

for $k \geq 4$ and

$$\mathcal{W}_{0,2} = \mathcal{W}_{2,0} = 0$$

for functions from $\underline{\mathcal{P}}$. The field component means (vacuum polarizations) are set to zero,

$$\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0,$$

for all components κ_1 without loss of generality since nonzero means are independently assigned as discussed in section 3.4.5. The remaining VEV is

$$\mathcal{W}_{0,0} := 1,$$

the normalization of the vacuum. In these examples, the elementary particles are created or annihilated in pairs although more generally, odd order VEV need not vanish in developments that include scalar fields [31, 33].

If the $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ in (46) are locally absolutely summable, polynomial growth functions of the momenta $\mathbf{p}_k$ after evaluation of the mass shell delta functions sets $p_{0k} = \pm\omega_k$, then the functions (46) are connected. Test functions provide that (45) is a summable function of $(\mathbf{p})_n$ that, due to translation invariance and after evaluation of the mass shell delta functions, includes a factor $\exp(i\rho\,\mathbf{a}\cdot(\mathbf{p}_1 + \ldots \mathbf{p}_k))$ with $1 < k < n-1$. Then, the Riemann-Lebesgue lemma and Poincaré invariance provide that the functions (46) decline as $\rho \to \infty$ for $a^2 < 0$ in (45).

3.4.2 Symmetric functions

In this section, notation is established to describe signed symmetric VEV functions. Local commutativity A.5 follows for signed symmetric VEV functions. The constructed VEV are signed symmetric with transpositions among the k $*$-dual function arguments or transpositions among the $n-k$ function arguments in the VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$, Argument type follows from the degenerate scalar product (21).

A convenient shorthand notation for VEV is to designate

$$\mathcal{W}_{k,n-k}(A) := \mathcal{W}_{k,n-k}((x)_A)_{(\kappa)_A} \tag{47}$$

for a set of n arguments with indices designated $A = \{i_1, \ldots i_n\}$. That is, $(x)_A := x_{i_1}, \ldots x_{i_n}$ and similarly $(\kappa)_A := \kappa_{i_1}, \ldots \kappa_{i_n}$.

In the notation (47), a sequence (24) of functions $\underline{\mathcal{V}}$ is considered *signed symmetric* if the elements satisfy

$$\mathcal{V}_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) = \sigma(\pi_1,(\kappa)_n)\sigma(\pi_2,(\kappa)_n)\mathcal{V}_{k,n-k}(\{1,n\}) \tag{48}$$

for every k, n. $\pi_1(\{1,n\})$ represents one of the $k!$ permutations of a sequence $\{1,k\}$. The permutation $\pi_1(\{1,n\})$ is applied to a sequence of n elements $\{1,n\}$. $\pi_2(\{1,n\})$ represents one of the $(n-k)!$ permutations of the sequence $\{k+1,n\}$ within a sequence of n elements $\{1,n\}$. That is, in a notation with permutations denoted

$$\pi_\nu(A) := \{\pi_\nu(i_1), \pi_\nu(i_2), \dots \pi_\nu(i_n)\},$$

and $\nu = 1, 2$,

$$\begin{aligned}\pi_1(\{1,n\}) &= \{\pi_1(1), \pi_1(2) \dots \pi_1(k), k+1, k+2, \dots n\}\\ \pi_2(\{1,n\}) &= \{1, 2, \dots k, \pi_2(k+1), \pi_2(k+2) \dots \pi_2(n)\}.\end{aligned}$$

Signs $\sigma(\pi_\nu,(\kappa)_n) = \pm 1$ are assigned to each permutation to satisfy local commutativity A.5 with normal commutation relations [9].

$$\begin{aligned}\sigma(\pi_1,(\kappa)_n) = 1 \qquad &\text{if} \qquad \pi_1(\{1,n\}) = \{1,n\}\\ \sigma(\pi_2,(\kappa)_n) = 1 \qquad &\text{if} \qquad \pi_2(\{1,n\}) = \{1,n\}.\end{aligned}$$

$$\sigma(\pi'_\nu,(\kappa)_n) = -\sigma(\pi_\nu,(\kappa)_n)$$

if the permutations π'_ν and π_ν differ by one transposition of two adjacent fermion indices, and

$$\sigma(\pi'_\nu,(\kappa)_n) = \sigma(\pi_\nu,(\kappa)_n),$$

if the transposition is either two adjacent boson indices or transposition of a boson with an adjacent fermion index. $\nu = 1, 2$. In (41) of section 3.3.1, arguments x_j, κ_j are assigned as boson or fermion by the value of $\kappa_j \in \{1, N_c\}$. The sign associated with an ordering of indices $A = \{i_1, i_2, \dots i_n\} = \pi_2(\pi_1(A_o))$ is with respect to a reference order $A_o := \{j_1, j_2, \dots j_n\}$. The sign for an A with respect to A_o is accumulated over a sequence of adjacent index transpositions that result in the index order A starting with A_o. The sign is determined by the product of signs from the sequence of transpositions, and the sign is independent of the choice for a sequence of transpositions that result in A from A_o [9].

The *signed symmetrization* $\underline{\mathcal{V}}$ of a sequence (24) of generalized functions $\underline{v}$ is

$$\mathcal{V}_{k,n-k}(A_o) := \sum_{\pi_1} \sigma(\pi_1,(\kappa)_{A_o}) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_{A_o})\, v_{k,n-k}(\pi_2(\pi_1(A_o))) \right). \tag{49}$$

The summations $\sum_{\pi_1}$ and $\sum_{\pi_2}$ include all $k!$ permutations of the indices labeled $\{1,k\}$ and all $(n-k)!$ permutations of the indices labeled $\{k+1,n\}$, respectively. The $\underline{\mathcal{V}}$ constructed from a generalized function $\underline{v}$ in (49) is signed symmetric.

The free field VEV are signed symmetric for function sequences from $\underline{\mathcal{P}}$. The expression (44) for $\underline{\mathcal{F}}$ is equivalent to

$$\mathcal{F}_{k,k}(\{1,2k\}) = \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_n) \frac{1}{k!} \prod_{j=1}^{k} W_2(x_j, x_{2k+1-j})_{\kappa_j \kappa_{k+j}} \right).$$

From the vanishing of two-point functions with indices of distinct type (41), the pair of indices of each contributing two-point function are necessarily the same type, and since $n = 2k$, transpositions among the last k indices are redundant with transpositions among the first k indices. The factor $k!$ compensates for the redundant transpositions. The sign is verified using (50) below and (6); the free field VEV are the signed symmetrization of a nonnegative form.

If $\underline{w}$ provides a degenerate scalar product, then its signed symmetrization $\underline{\mathcal{W}}$ also provides a degenerate scalar product since

$$\mathcal{W}(\underline{f}^*, \underline{f}) = w(\underline{g}^*, \underline{g}) \geq 0 \tag{50}$$

with

$$\underline{g} = (f_0, f_1, \ldots \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) f_k(\pi_1(\{1,k\})), \ldots)$$

from (49) for the scalar product (21).

If $\underline{\mathcal{V}}$ provides a degenerate scalar product, then a *scaled sequence* of functions $\underline{\mathcal{V}_a}$ generated from $\underline{\mathcal{V}}$ as

$$\underline{\mathcal{V}_a} := (|a_0|^2, \overline{a}_1 a_0 \mathcal{V}_{1,0}, \ldots \overline{a}_k a_{n-k} \mathcal{V}_{k,n-k}, \ldots), \tag{51}$$

also provides a degenerate scalar product. Each $a_k \in \mathbb{C}$ and $|a_k| \neq 0$. That $\mathcal{V}_a(\underline{f}^*, \underline{f}) \geq 0$ follows from $\mathcal{V}(\underline{g}^*, \underline{g}) \geq 0$ by setting the sequence $\underline{f} = (g_0/a_0, g_1/a_1, \ldots) \in \underline{\mathcal{P}}$ in the scalar product (21).

3.4.3 Functions that exhibit interaction

The higher order connected functions are conveniently distinguished with the designation ${}^C\underline{\mathcal{U}}$.

$${}^C\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n} := {}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \qquad \text{if} \quad n \geq 4 \tag{52}$$

with ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ from the description of connected functions (46) and the ${}^C\mathcal{U}_{k,n-k}$ are zero for $n < 4$. From (46), the Fourier transforms of the connected functions are

$${}^C\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n} := c_n \delta(p_1 + \ldots p_n)\, Q_{k,n-k}((p)_n)_{(\kappa)_n}.$$

These $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ are Lorentz covariant, are supported solely on mass shells, and consist of summations with nonnegative weight of factors common across orders k, n analogously to the

factors of positive matrices [27] or nonnegative measures from the Bochner–Schwartz theorem for positive distributions [21]. Translation invariance of the scalar product (21) is equivalent to energy-momentum conservation. Limiting the momentum support of the connected ${}^{C}\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n}$ to mass shells associates states with free particles described by representations of the Lorentz group [9]. From cluster decomposition A.6, function sequences $\underline{f} \in \mathbf{H}_{\mathcal{P}}$ describe nearly free particles when the spatial support of each argument is isolated from the spatial support of all other arguments.

With the substitutions,

$$\delta(p_1 + \dots p_n) = \int \frac{du}{(2\pi)^4} \prod_{j=1}^{n} e^{-ip_j u}$$

and

$$c_n := \int d\sigma(\lambda)\ \lambda^n, \tag{53}$$

the Fourier transforms of the ${}^{C}\mathcal{U}_{k,2n-k}((x)_{2n})_{(\kappa)_{2n}}$ are

$${}^{C}\widetilde{\mathcal{U}}_{k,2n-k}(\{1,2n\}) := \iint d\sigma(\lambda)\frac{du}{(2\pi)^4}\ \lambda^{2n} \prod_{j=1}^{2n} e^{-ip_j u}\ Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}. \tag{54}$$

using the shorthand notation (47). The summation du is over $\mathbb{R}^4$ and $d\sigma(\lambda)$ is a summation with nonnegative weight over $\mathbb{R}$. Then $c_{2n} \geq 0$. For example, $c_n = \rho^n$ for $d\sigma(\lambda) = \delta(\lambda - \rho)d\lambda$, or $c_n = n!/\rho^{n+1}$ for $\theta(\lambda)\exp(-\rho\lambda)d\lambda$ with $0 < \rho \in \mathbb{R}$. The form (54) is selected to factor ${}^{C}\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$ and demonstrate nonnegativity of the scalar product (21).

The array $Q_{k,n-k}((p)_n)_{(\kappa)_n}$ is signed symmetric (49) with terms that factor appropriately for nonnegativity of the scalar product.

$$\begin{aligned} Q_{k,n-k}((p)_n)_{(\kappa)_n} &:= \sum_{\pi_2} \sigma(\pi_2, (\kappa)_n) \left(\sum_{\pi_1} \sigma(\pi_1, (\kappa)_n)\ q_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) \right) \\ q_{k,n-k}((p)_n)_{(\kappa)_n} &:= \prod_{j=1}^{n} \left(\delta(p_j^2 - \lambda_{cj}^{-2}) \frac{d}{d\rho_j} \right) \exp\Big(\sum_{a,b \in \mathbb{J}_{k,n}} \rho_a \rho_b H(p_a, p_b)_{\kappa_a \kappa_b} \Big) \end{aligned} \tag{55}$$

with the $q_{k,n-k}((p)_n)_{(\kappa)_n}$ evaluated at $(\rho)_n = 0$ after the differentiations. The λ_{cj} are the Compton wavelengths (14) for the finite masses m_{κ_j}. To comply with elemental stability A.7,

$$Q_{k,2n-k}(\{1,2n\}) := 0 \qquad \text{if} \quad k = 0, 1, 2n-1, 2n.$$

Justified in (28), ${}^{C}\mathcal{U}_{k,n-k}$ is limited to $n \geq 4$ to eliminate a divergent two-point function that results from substitution of the extrapolation of (55) to $n = 2$ into (54). The exponential function in $q_{k,n-k}((p)_n)_{(\kappa)_n}$ is a function of products of pairs $\rho_a \rho_b$ with $a \neq b$ and as a consequence

of evaluation at $(\rho)_n = 0$, the odd order connected functions are zero.

$$Q_{k,2n+1-k}(\{1, 2n+1\}) = 0.$$

The summations over argument transpositions π_1 and π_2 are described in section 3.4.2. Signed symmetry (48) of ${}^C\mathcal{U}_{k,n-k}(\{1,2n\})$ follows from the signed symmetry of $Q_{k,n-k}((p)_n)_{(\kappa)_n}$. The $H(p_a, p_b)_{\kappa_a\kappa_b}$ are functions defined below in (56) and the summation $a, b \in \mathbb{J}_{k,n}$ with $a, b \in \{1, n\} \subset \mathbb{N}$ is defined below in (58). These forms are selected to factor the constructed ${}^C\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$.

The $H_{\kappa_a\kappa_b}$ are $N_c \times N_c$ matrices of generalized functions over the energy-momenta.

$$H(p_a, p_b)_{\kappa_a\kappa_b} := \begin{cases} B(p_a + p_b)_{\kappa_a\kappa_b} & \text{if } a, b \in \{1, k\} \text{ or } a, b \in \{k+1, n\} \\ \Upsilon(-p_a + p_b)_{\kappa_a\kappa_b} & \text{if } a \in \{1, k\}, b \in \{k+1, n\} \\ 0 & \text{if } a \in \{k+1, n\}, b \in \{1, k\}. \end{cases} \tag{56}$$

That is, $H = B$ if both arguments are $*$-dual function arguments or if both arguments are function arguments, and $H = \Upsilon$ or zero if the arguments are split, one $*$-dual function and one function argument in the scalar product (21). Addressed in section 3.5.5, this form for $H(p_1, p_2)$ satisfies regularity A.1 with nonconstant $H(p_1, p_2)$. Due to Lorentz covariance, nonconstant $H(p_1, p_2)$ are necessarily singular within $\mathbb{R}^8$ and (56) is selected to exclude the singularities from $\mathcal{E}_n^+$, the region (36) of joint support of ${}^C\widetilde{\mathcal{U}}_{k,n-k}((p)_n)_{(\kappa)_n}$ and functions from $\underline{\mathcal{P}}$. The matrices $B(p)$ and $\Upsilon(p)$ are Fourier or Laplace transforms of the two-point function array $M(p)$ in the two-point function (37) in the connected functions (46).

$$\begin{aligned} B(p) &:= \int d\mu_B(s)\, e^{isp} M(s) \\ \Upsilon(p) &:= \int d\mu_\Upsilon(s)\, e^{-sp} M(s) \end{aligned} \tag{57}$$

with $s \in \mathbb{R}^4$, $d\mu_B(s)$ and $d\mu_\Upsilon(s)$ are real-valued Lorentz invariant measures, $d\mu_\Upsilon(s)$ is a nonnegative measure, and $sp = s_0 p_0 - \mathbf{s} \cdot \mathbf{p}$ is Lorentz invariant. The forms (57) with the factorization (39) of $M(p)$ complete a factorization of the constructed ${}^C\mathcal{U}_{k,n-k}((x)_n)_{(\kappa)_n}$. Considering the applications of D from the $*$-dual functions (8), three sets of factors are distinguished in (55).

$$\begin{aligned} (D\cdot)_k \exp\Big(\sum_{a,b\in\mathbb{J}_{k,n}} \rho_a\rho_b H_{\kappa_a\kappa_b}\Big) &:= \exp\Big(\sum_{b_1>a_1=1}^{k} \rho_{a_1}\rho_{b_1}(DBD^T)_{\kappa_{a_1}\kappa_{b_1}}\Big) \\ &\times \exp\Big(\sum_{b_2>a_2=k+1}^{n} \rho_{a_2}\rho_{b_2} B_{\kappa_{a_2}\kappa_{b_2}}\Big)\ \exp\Big(\sum_{a_3=1}^{k}\sum_{b_3=k+1}^{n} \rho_{a_3}\rho_{b_3}(D\Upsilon)_{\kappa_{a_3}\kappa_{b_3}}\Big). \end{aligned} \tag{58}$$

$\sum_{b>a=1}^{k}$ designates $\sum_{a=1}^{k-1}\sum_{b=a+1}^{k}$. (58) defines the summation $a, b \in \mathbb{J}_{k,n}$ in (55). Many other forms for $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ also provide realizations that satisfy A.1-7. For example, the substitution

$$\sum_{a_3=1}^{k}\sum_{b_3=k+1}^{n}\rho_{a_3}\rho_{b_3}(D\Upsilon)_{\kappa_{a_3}\kappa_{b_3}} \mapsto \sum_{a_3=1}^{k}\sum_{b_3=k+1}^{n}\overline{z}_{k+1-a_3}z_{b_3-k}\,\rho_{a_3}\rho_{b_3}(D\Upsilon)_{\kappa_{a_3}\kappa_{b_3}}$$

with $z_j \in \mathbb{C}$ can provide a contribution from Υ for fermions. Demonstration that the factorization of ${}^{C}\underline{\mathcal{U}}$ provides a degenerate scalar product (21) is in section 3.5.2.

Like $M(p)$, the $H(p_a, p_b)_{\kappa_a\kappa_b}$ described in (56) and (57) are zero if κ_a and κ_b are distinct index types, a fermion with a boson index.

The elements $M(p)_{\kappa_a\kappa_b}$ are multinomials in the energy-momentum components and greater spins result in higher order multinomials [9]. The selected support of the VEV and functions in $\underline{\mathcal{P}}$ satisfy regularity A.1 by appropriate exclusion of points $(p_a \pm p_b)^2 = 0$. From the limitation of the energy-momentum support of the VEV and functions (11) in $\underline{\mathcal{P}}$ to $\mathcal{E}_n^+$, the singularities of $H(p_1, p_2)$ at $(p_a \pm p_b)^2 = 0$ are excluded from the summations in the scalar products (21). The indicated dependence on $M(p)_{\kappa_1\kappa_2}$ and vanishing of the odd orders $\mathcal{U}_{k,2n-1-k}$ implements Poincaré invariance of the scalar product for nonzero spins, [37] and section 3.5.3. Odd orders can be included for spin zero fields [33].

The expansions (54), (55) and (58) complete construction of example physically nontrivial higher order connected functions. The constructed ${}^{C}\underline{\mathcal{U}}$ are connected (45) and signed symmetric (48).

3.4.4 Cluster expansion: composition of connected functions into VEV

In this section, the cluster expansion of a generalized function sequence $\underline{\mathcal{W}}$ is described. The VEV (26) are composed of the connected functions (46) described in sections 3.3.1 and 3.4.3.

Cluster expansion is illustrated by the two- and four-point VEV functions. If the permutation matrix O in (41) is the identity, with designations B for the boson and F for the fermion blocks of the block diagonal ${}^{C}\mathcal{W}_{1,1}$, and in the abbreviated notation (47),

$$\begin{aligned} B(12) &:= {}^{C}\mathcal{W}_{1,1}(12) && \text{if } 1 \le \kappa_1, \kappa_2 \le N_b \\ F(12) &:= {}^{C}\mathcal{W}_{1,1}(12) && \text{if } N_b + 1 \le \kappa_1, \kappa_2 \le N_c \\ {}^{C}\mathcal{W}_{1,1}(12) &= 0 && \text{if the index types of } \kappa_1 \text{ and } \kappa_2 \text{ differ,} \end{aligned}$$

then the cluster expansion is

$$\begin{aligned} \mathcal{W}_{1,1}(12) &= B(12) + F(12) \\ \mathcal{W}_{2,2}(1234) &= B(13)B(24) + B(14)B(23) \\ &\quad + B(14)F(23) + B(13)F(24) + F(13)B(24) + F(14)B(23) \\ &\quad - F(13)F(24) + F(14)F(23) + {}^{C}\mathcal{W}_{2,2}(1234). \end{aligned} \tag{59}$$

In (59), vacuum polarization is zero. N_b is the number of boson field components and $N_b \leq N_c$, the number of field components. For the energy support limited functions (11) from $\underline{\mathcal{P}}$, only $\mathcal{W}_{1,1}(f_1^* \, g_1)$ contributes.

$$\mathcal{W}_{1,1}(f_2) = \mathcal{W}_{1,1}(f_2^*) = \mathcal{W}_{1,1}(f_1 \, g_1^*) = 0$$

if $f_2, f_1, g_1 \in \underline{\mathcal{P}}$. The index type of each index κ_j, boson or fermion, is determined in (41) and the free field contribution to (59) is determined in section 3.3.1 by the commutation/anti-commutation relations of free field operators. The composition (59) satisfies both local commutativity A.5 with normal commutation relations [9], and cluster decomposition A.6. ${}^C\mathcal{W}_{2,2}$ is signed symmetric (48). Satisfaction of local commutativity A.5 follows from the signed symmetry of the connected functions for function sequences from $\underline{\mathcal{P}}$.

$$\begin{aligned} \mathcal{W}_{2,2}(1234) &= \pm\mathcal{W}_{2,2}(2134) \\ &= \pm\mathcal{W}_{2,2}(1243) \end{aligned}$$

where $\pm$ is the sign from transposition of the arguments x_1, κ_1 with x_2, κ_2, or from the transposition of the arguments x_3, κ_3 with x_4, κ_4, respectively. The strong cluster decomposition condition A.6 is satisfied due to the connectivity (45) of the ${}^C\mathcal{W}_{k,n-k}$. For arguments 1 and 3 space-like distant from arguments 2 and 4 (two $*$-dual arguments, two normal arguments, the support of the functions $g_1^* f_1^* f_1 g_1$ has the support of the f's arbitrarily space-like distant from the support of the g's),

$$\mathcal{W}_{2,2}(1234) \to \mathcal{W}_{1,1}(14)\mathcal{W}_{1,1}(23).$$

The remaining cases in axiom A.6 ($f_2^* g_2$, $f_2^* f_1 g_1$ and $f_1^* g_1^* f_2$) all result in the four-point functions asymptotically approaching zero.

More generally, VEV are determined by the sequence of functions $\underline{\mathcal{W}}$ expanded in signed products of free field VEV functions $\underline{\mathcal{F}}$ with VEV functions $\underline{\mathcal{U}}$ that introduce interaction.

$$\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}. \tag{60}$$

$\underline{\mathcal{F}}$ from (44) results from cluster expansion of the two-point connected function and $\underline{\mathcal{U}}$ results from cluster expansion of the sequence of connected functions ${}^C\underline{\mathcal{U}}$ described in section 3.4.3. To satisfy Poincaré invariance A.3, the sequences $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$ composed in the construction (60) transform with the same representation of the Lorentz subgroup,

A signed product of two VEV function sequences (24) $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ is designated the $\circ$-product. The resulting signed symmetric (48) VEV function sequence $\underline{\mathcal{W}}$ is described

$$\underline{\mathcal{W}} := \underline{\mathcal{T}} \circ \underline{\mathcal{V}} \tag{61}$$

with elements that are the signed symmetrization (49) of a VEV function sequence $\underline{w}$ constructed from the elements of $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$.

$$\begin{aligned} \mathcal{W}_{k,n-k}(\{1,n\}) &= \sum_{\pi_1} \sigma(\pi_1,(\kappa)_n) \left(\sum_{\pi_2} \sigma(\pi_2,(\kappa)_n)\; w_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) \right) \\ w_{k,n-k}(\{1,n\}) &:= \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell!j!} \; \frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!} \end{aligned} \tag{62}$$

with the reference argument argument order A_o, A'_o

$$\begin{aligned} A_o &:= \{1,\ell\} \cup \{k+1, k+j\} \\ A'_o &= \{1,n\} \setminus A_o = \{\ell+1, k\} \cup \{k+j+1, n\}. \end{aligned} \tag{63}$$

The terms in the summation (62) with the arguments in the reference order have positive sign. In section 3.5.1 it is demonstrated that this choice of sign results in a nonnegative degenerate scalar product (21). From (26), $*$-dual arguments are antiordered in the scalar product but arguments of the VEV functions $\underline{\mathcal{W}}$ are in natural order. Signs are determined in section 3.4.2 to exhibit normal statistics. $\ell = 0$ provides that there are no $*$-dual arguments in $\mathcal{T}_{0,j}$ and $\ell = j$ provides that there are no function arguments in $\mathcal{T}_{j,j}$. Similarly for $\mathcal{V}_{k-\ell,n-k-j}$ if $\ell = k$ or $\ell = 2k + j - n$. The terms in the expansion $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ are products of elements from $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$, factors include no arguments in common and all arguments are present in each term of the expansion (62). As a consequence, the sequence $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ consists of generalized functions if $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are sequences of generalized functions.

For signed symmetric factors $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ in a $\circ$-product, the assignment of signs (48),—

$$\mathcal{V}_{k,n-k}(\pi_2(\pi_1(\{1,n\}))) = \sigma(\pi_1,(\kappa)_n)\sigma(\pi_2,(\kappa)_n)\mathcal{V}_{k,n-k}(\{1,n\})$$

and similarly for $\mathcal{T}_{k,n-k}$, enables a convenient form for the elements of $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ with the arguments in natural order. Use of (48) to transpose the arguments into natural order,

$$\mathcal{W}_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k} \sum_{j=0}^{n-k} \left(\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2} \sigma_1\sigma_2 \; \mathcal{T}_{\ell,j}(A) \; \mathcal{V}_{k-\ell,n-k-j}(A') \right) \tag{64}$$

with the abbreviated notation $\sigma_\nu := \sigma(\pi_\nu,(\kappa)_n)$ for $\nu = 1, 2$, and

$$A = \pi_1(\pi_2(A_o)) \quad \text{and} \quad A' = \pi_1(\pi_2(A'_o))$$

with the reference order A_o and A'_o from (63). The A' are the set complements with respect to $\{1, n\}$ of A. The summations $\sum_{\mathfrak{s}_1} \sum_{\mathfrak{s}_2}$ include only the subset of the permutations π_1, π_2,

$$\{\mathfrak{s}_\nu\} \subset \{\pi_\nu\}$$

that result in naturally ordered A and A'. With

$$A = \pi_1(\pi_2(A_o)) = i_1, i_2, \ldots i_{\ell+j},$$

the permutation is included in the summation $\sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}$ only if $i_1 < i_2 < \ldots i_{\ell+j}$ and the $n-\ell-j$ argument indices in A' are similarly ordered. Equality of (64) with (62) follows from the signed symmetry of $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ that preserves the terms in (62) with the selection of argument order. After the indicated argument reorderings, there are $\ell!j!(k-\ell)!(n-k-j)!$ of each distinct term. The normalization in (62) results in a magnitude unity coefficient for each term $\mathcal{T}_{\ell,j}(A)\,\mathcal{V}_{k-\ell,n-k-j}(A')$ with distinct, naturally ordered sets of arguments.

Demonstrated in section 3.5, axioms A.1-7 are satisfied if the factor sequences in the construction (60) are signed symmetric (48) and each factor sequence satisfies A.1-7. The free field contributions $\underline{\mathcal{F}}$ associate the constructions with particles, section 3.6, and the higher-order connected functions $\underline{\mathcal{U}}$ introduce interaction, sections 3.9 and 4. In the construction (60) of VEV, $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$, only the $\mathcal{F}_{\ell,\ell}$ contribute for function sequences from $\underline{\mathcal{P}}$. Consequently, only terms with $j = \ell$ contribute to the summation in (62) and then

$$w_{k,n-k}(\{1,n\}) := \sum_{\ell=0}^{k} \frac{\mathcal{F}_{\ell,\ell}(A_o)}{(\ell!)^2}\,\frac{\mathcal{U}_{k-\ell,n-k-\ell}(A'_o)}{(k-\ell)!(n-k-\ell)!}. \tag{65}$$

The $\circ$-product (62) is commutative, associative and distributive with addition for VEV function sequences. section 3.5.3. Relabeling the summations in the $\circ$-product (62) $\ell' = k-\ell$ and $j' = n-k-j$,

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell=0}^{k}\sum_{j=0}^{n-k} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell!j!}\,\frac{\mathcal{V}_{k-\ell,n-k-j}(A'_o)}{(k-\ell)!(n-k-j)!}$$

becomes

$$w_{k,n-k}(\{1,n\}) = \sum_{\ell'=0}^{k}\sum_{j'=0}^{n-k} \frac{\mathcal{V}_{\ell',j'}(A'_o)}{\ell'!j'!}\,\frac{\mathcal{T}_{k-\ell',n-k-j'}(A_o)}{(k-\ell')!(n-k-j')!}.$$

This second expression is (62) for $\underline{\mathcal{V}} \circ \underline{\mathcal{T}}$ with the same terms and signs as $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$. Then, $\underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ is commutative. That (62) is a summation of arithmetic multiplications provides that the $\circ$-product is associative and distributive with addition of sequences.

The sequence

$$\exp\circ(\underline{\mathcal{T}}) := \Omega + \sum_{j=1}^{\infty} \frac{1}{j!}\,\underline{\mathcal{T}} \circ \underline{\mathcal{T}} \circ \ldots \underline{\mathcal{T}} \tag{66}$$

has j factors of $\underline{\mathcal{T}}$ in the jth term and the $\circ$-product is from (61). Ω is the sequence $(1,0,0,\ldots)$. $\exp\circ(\underline{\mathcal{T}})$ is designated the cluster expansion of $\underline{\mathcal{T}}$.

The sequence $\underline{\mathcal{U}}$ is the cluster expansions (66) of the higher order connected functions (54),

$$\underline{\mathcal{U}} := \exp\circ({}^C\underline{\mathcal{U}}). \tag{67}$$

The free field functions $\underline{\mathcal{F}}$ in (44) constructed from the commutation and vacuum functional properties of the algebra of Hermitian free field operators are also a cluster expansion. The sequence

$$\underline{\mathcal{F}} = \exp\circ({}^C\underline{\mathcal{F}}) \tag{68}$$

defined with

$${}^C\mathcal{F}_{1,1} := W_2.$$

W_2 is the two-point function from (46) and all other ${}^C\mathcal{F}_{k,n-k} = 0$. The nonzero elements of $\underline{\mathcal{F}}$ are

$$\mathcal{F}_{k,k} = \frac{1}{k!}({}^C\underline{\mathcal{F}}\circ)^k.$$

Equality of the cluster expansion with the algebraic construction of $\underline{\mathcal{F}}$ in (44) is demonstrated by induction. Inspection of (44) and (68) verifies agreement for $k = 1$. Assertion that

$$\mathcal{F}_{k,k}(\{1,2k\}) = \sum_S \sigma(S,(\kappa)_{2k}) \prod_{j=1}^k W_2(j,i_j)$$

and application of the cluster expansion to evaluate the next element provides

$$\begin{aligned}\mathcal{F}_{k+1,k+1}(\{1,2k+2\}) &= \frac{1}{k+1}\,\mathcal{F}_{k,k}(\{1,k\}\cup\{k+2,2k+1\})\circ W_2(k+1,2k+2)\\ &= \frac{1}{k+1}\sum_S \sigma_S \prod_{j=1}^k W_2(j,i_j)\circ W_2(k+1,2k+2)\end{aligned}$$

with notation from (44), (47) and (66), and the function argument labels $\{i_j\}$ adjusted to $i_j \in \{k+2,2k+1\}$ for $j \in \{1,k\}$.

$$\sigma_S := \sigma(S,\{1,k\}\cup\{k+2,2k+1\})$$

from the algebraic construction (44). Substitution of the $\circ$-product (64) provides

$$\mathcal{F}_{k+1,k+1}(\{1,2k+2\}) = \sum_{\mathfrak{s}_1}\sum_{\mathfrak{s}_2}\frac{\sigma_1\sigma_2}{k+1}\left(\sum_S \sigma_S \prod_{j=1}^k W_2(\mathfrak{s}_2(\mathfrak{s}_1(j)),\mathfrak{s}_2(\mathfrak{s}_1(i_j)))W_2(\mathfrak{s}_2(\mathfrak{s}_1(k+1)),\mathfrak{s}_2(\mathfrak{s}_1(2k+2)))\right)$$

with the abbreviated notation $\sigma_\nu := \sigma(\pi_\nu,(\kappa)_{2k+2})$ for $\nu = 1,2$. The sums over permutations $\mathfrak{s}_1,\mathfrak{s}_2$ are from (64). From the free field sequence $\underline{\mathcal{F}}$ in the algebraic construction (44), the

summation $\sum_S$ includes the $k!$ distinct pairings j, i_j with $j \in \{1, k\}$ and $i_j \in \{k+2, 2k+1\}$. $S = \{1, \ldots k, i_1, \ldots i_k\}$. From section 3.4.2, signs accumulate as the product of signs from adjacent argument transpositions that accumulate to achieve a final argument order. Then,

$$\sigma(S', (\kappa)_{2k+2}) = \sigma(\pi_1, (\kappa)_{2k+2})\sigma(\pi_2, (\kappa)_{2k+2})\sigma(S, (\kappa)_{2k})$$

with $S' = \{1, \ldots k+1, i_1, \ldots i_k, 2k+2\}$ with reference argument order $\sigma(S'_o) = 1$ for $S'_o = \{k+1, \ldots 1, k+2, \ldots 2k+2\}$. The terms are each a product of identical two-point functions with distinct pairings of arguments, and the indicated permutations $\mathfrak{s}_1, \mathfrak{s}_2$ result in an accumulation of $k+1$ of each distinct term. Then

$$\mathcal{F}_{k+1,k+1}(\{1, 2k+2\}) = \sum_{S'} \sigma_{S'} \prod_{j=1}^{k+1} W_2(j, i_j)$$

with summation over distinct pairings of arguments. Finally, induction verifies that the two expressions for the free field VEV are equivalent for function sequences from $\underline{\mathcal{P}}$.

From (66), the commutivity, associativity and distributivity with addition of sequences of the $\circ$-product (61), and the binomial expansion provide

$$\begin{aligned} \exp\circ(\underline{\mathcal{T}} + \underline{\mathcal{V}}) &= \sum_{N=0}^{\infty} \frac{1}{N!} \left((\underline{\mathcal{T}} + \underline{\mathcal{V}})\circ\right)^N \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\nu=0}^{N} \binom{N}{\nu} (\underline{\mathcal{T}}\circ)^\nu \circ (\underline{\mathcal{V}}\circ)^{N-\nu} \\ &= \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \frac{1}{N_1! N_2!} (\underline{\mathcal{T}}\circ)^{N_1} \circ (\underline{\mathcal{V}}\circ)^{N_2} \\ &= (\exp\circ(\underline{\mathcal{T}})) \circ (\exp\circ(\underline{\mathcal{V}})) \end{aligned} \tag{69}$$

from relabeling terms in summations. The expansions for $\underline{\mathcal{U}}$ and $\underline{\mathcal{F}}$, (67) and (68) respectively, substituted into the construction (60) provides that

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^C\underline{\mathcal{F}}) \circ \exp\circ({}^C\underline{\mathcal{U}}) \\ &= \exp\circ({}^C\underline{\mathcal{F}} + {}^C\underline{\mathcal{U}}) \end{aligned} \tag{70}$$

from the identity (69). The sequences of connected functions add in the construction (60).

Inversion of (66) defines truncated functions ${}^T\underline{\mathcal{W}}$ from a sequence $\underline{\mathcal{W}}$.

$$\underline{\mathcal{W}} := \exp\circ({}^T\underline{\mathcal{W}}). \tag{71}$$

Similarly to RQFT [9], truncated functions are connected in the constructions if vacuum polarization is zero. If both ${}^1\underline{\mathcal{W}}$ and ${}^2\underline{\mathcal{W}}$ are signed symmetric and satisfy A.1-7, then ${}^1\underline{\mathcal{W}}\circ{}^2\underline{\mathcal{W}}$ realizes relativistic quantum physics and this leads to consideration of prime and divisible realizations [26]. From (69), truncated functions add.

The cluster expansions (67) and (68) and specification of the connected functions (46) complete these contructions of physically nontrivial realizations of RQP. There is a realization of relativistic quantum physics for: every realization of free fields (selection of $N_c, M(p), D, S(A)$); nonnegative measure $d\sigma$ (that determines c_n, the relative contributions of the connected functions); and Lorentz invariant measures $d\mu_B(p)$ and $d\mu_\Upsilon(p)$ (that characterize interactions). Satisfaction of prospective axioms A.1-7 is verified in section 3.5.

3.4.5 Vacuum polarizations

Mean VEV of the field $\langle\Omega|\Phi(x)_\kappa\Omega\rangle$ are set to zero in the development above without loss of generality. Nonzero means are independently specified for the boson field components without impact to satisfaction of A.1-7. The addition of a constant to the quantum field defined by (25) implements fields $\Phi(x)_\kappa$ with finite means except, if κ is a fermion index, then addition of a constant to the field violates anticommutation.

The VEV

$$(D\cdot)_k\,\mathcal{V}_{k,n-k}(\{1,n\}) := \prod_{\ell=1}^{k}\overline{a}_{\kappa_\ell}\prod_{j=k+1}^{n}a_{\kappa_j}$$

satisfy A.1-6 and an appropriately restated A.7. The constants

$$a_\kappa := \langle\Omega|\Phi(x)_\kappa\Omega\rangle$$

and a_κ is finite only for boson indices κ. Applications of Dirac conjugations D use the matrix notation (9) and the abbreviated notation (47) is used for the VEV functions.

$\underline{\mathcal{V}}$ is signed symmetric and as a consequence of the properties of the $\circ$-product (61), and from (71),

$$\underline{\mathcal{W}} = \underline{\mathcal{F}}\circ\underline{\mathcal{U}}\circ\underline{\mathcal{V}} = \exp\circ({}^C\underline{\mathcal{F}} + {}^C\underline{\mathcal{U}} + {}^T\underline{\mathcal{V}})$$

satisfies A.1-6 and an appropriately restated A.7.

3.5 Satisfaction of the axioms

3.5.1 Positive definite $\circ$-products

In this section, it is demonstrated that the $\circ$-product composition of two signed symmetic VEV function sequences (24) that each provide a degenerate scalar product (21) provides a degenerate scalar product. From (62), $\underline{\mathcal{W}} = \underline{\mathcal{T}}\circ\underline{\mathcal{V}}$ is the signed symmetrization (49) of $\underline{w}$. From (50), if $\underline{w}$ provides a degenerate scalar product then $\underline{\mathcal{W}}$ provides a degenerate scalar product.

In this section it is demonstrated that $w(\underline{f}^*, \underline{g})$ with the scalar product (21) and $\underline{w}$ from (62) provides a degenerate scalar product for elements in the tensor product of linear vector spaces each with degenerate scalar products from $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$, respectively. Degenerate scalar products $\underline{\mathcal{T}_a}$ and $\underline{\mathcal{V}_a}$ result from scaled sequences of the degenerate scalar products $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ using (51). Linear vector spaces with degenerate scalar products are also designated *pre-Hilbert spaces*.

If both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ provide degenerate scalar products, then the tensor product of the pre-Hilbert spaces that result from scaled $\underline{\mathcal{V}}$ and $\underline{\mathcal{T}}$ has elements labeled $\underline{f} \otimes \underline{g}$ and a degenerate scalar product

$$w((\underline{f}_1 \otimes \underline{f}_2)^*, \underline{g}_1 \otimes \underline{g}_2) = \mathcal{T}_a(\underline{f}_1^*, \underline{g}_1)\, \mathcal{V}_a(\underline{f}_2^*, \underline{g}_2). \tag{72}$$

The scales (51) on the degenerate scalar products are $a_\ell := 1/\ell!$ in both instances. In the pre-Hilbert spaces of function sequences based on $\underline{\mathcal{P}}$ for generalized functions $\underline{\mathcal{T}_a}$ and $\underline{\mathcal{V}_a}$, fields $\Phi_\mathcal{T}$ and $\Phi_\mathcal{V}$ are defined (25). These operations extend to the tensor product space. The elevations $\Phi_\mathcal{T} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_\mathcal{V}$ to the tensor product space are

$$(\Phi_\mathcal{T} \otimes \mathbb{I})\underline{f} \otimes \underline{g} = (\Phi_\mathcal{T}\underline{f}) \otimes \underline{g}$$

and

$$(\mathbb{I} \otimes \Phi_\mathcal{V})\underline{f} \otimes \underline{g} = \underline{f} \otimes (\Phi_\mathcal{V}\underline{g}).$$

The composite vacuum is $\Omega = \Omega_\mathcal{T} \otimes \Omega_\mathcal{V}$. To compact notation, the arguments of the fields are understood. The construction is a variation of a construction due to Borchers and Uhlmann [26, 47]. $|O(f_n)\Omega\rangle$ with

$$O(f_n) := \sum_{(\kappa)_n} \int d(x)_n \left(\sum_{\ell=0}^{n} \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!} \right) f_n((x)_n)_{(\kappa)_n}$$

are elements of the tensor product space with

$$(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell} := \left(\prod_{j=1}^{\ell} (\Phi_\mathcal{T}(x_j)_{\kappa_j} \otimes \mathbb{I}) \right) \left(\prod_{j=\ell+1}^{n} (\mathbb{I} \otimes \Phi_\mathcal{V}(x_j)_{\kappa_j}) \right).$$

In the tensor product space, $\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle$ is the degenerate scalar product for function sequences from $\underline{\mathcal{P}}$.

$$\begin{aligned} \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega \rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \\ \times \langle \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^\ell (\mathbb{I} \otimes \Phi_\mathcal{V})^{n-\ell}}{\ell!(n-\ell)!}\, \Omega | \frac{(\Phi_\mathcal{T} \otimes \mathbb{I})^j (\mathbb{I} \otimes \Phi_\mathcal{V})^{m-j}}{j!(m-j)!}\, \Omega \rangle \\ \times \overline{f_n((x)_n)}_{(\kappa)_n}\, g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}. \end{aligned}$$

The definition of scalar product, and the operators $\Phi_{\mathcal{T}}(x_j)_{\kappa_j} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \Phi_{\mathcal{V}}(x_j)_{\kappa_j}$, the $*$-dual (8) and the relation between VEV and generalized functions (26) result in

$$\sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle = \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(x)_{n+m} \sum_{\ell=0}^{n} \sum_{j=0}^{m} \frac{\mathcal{T}_{\ell,j}(A_o)}{\ell! j!} \frac{\mathcal{V}_{n-\ell,m-j}(A'_o)}{(n-\ell)!(m-j)!} \times f_n((x)_n)^*_{(\kappa)_n} g_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

for A_o, A'_o from (63). Then substitution of the expression (62) for functions $w_{k,n-k}$ identifies that

$$w(\underline{f}^*, \underline{g}) = \sum_{n,m} \langle O(f_n)\Omega | O(g_m)\Omega\rangle. \tag{73}$$

As a consequence. $w(\underline{f}^*, \underline{f}) \geq 0$ since it is the squared norm of a vector in the tensor product space.

Satisfaction of positive definiteness A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{T}} \circ \underline{\mathcal{V}}$ follows from (50), (51) and (62) if both $\underline{\mathcal{T}}$ and $\underline{\mathcal{V}}$ are signed symmetric and satisfy A.1-2 for function sequences $\underline{\mathcal{P}}$. For $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$,

$$\mathcal{W}(\underline{f}^*, \underline{f}) \geq 0$$

if $\underline{\mathcal{U}}$ satisfies positive definiteness since it is established that $\underline{\mathcal{F}}$ is positive, appendix 6.9. $\underline{\mathcal{U}}$ is constructed as signed symmetric and from (44), $\underline{\mathcal{F}}$ is signed symmetric. Satisfaction of positive definiteness A.2 for $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ reduces to demonstration that $\underline{\mathcal{U}}$ satisfies positive definiteness.

3.5.2 Positive definiteness of $\underline{\mathcal{U}}$

In this section it is demonstrated that the constructed signed symmetric sequences (24) of generalized functions $\underline{\mathcal{U}}$ from (67) with (54) provide degenerate scalar products (21) for $\underline{\mathcal{P}}$ and thus satisfy positive definiteness A.2.

The construction (67) of $\underline{\mathcal{U}}$ as $\exp \circ ({}^{C}\underline{\mathcal{U}})$ and that a positively weighted summation of generalized functions that provide degenerate scalar products provides a degenerate scalar product,

$$(a\mathcal{T} + b\,\mathcal{V})(\underline{f}^*, \underline{f}) = a\mathcal{T}(\underline{f}^*, \underline{f}) + b\,\mathcal{V}(\underline{f}^*, \underline{f}) \geq 0$$

if $a, b > 0$, section 3.5.1, provides that the scalar product from $\underline{\mathcal{U}}$ satisfies positive definiteness if the scalar products from $({}^{C}\underline{\mathcal{U}}\circ)^{\ell}$ are positive definite. Then, since the $\circ$-product preserves signed symmetry, section 3.4.4, and the $\circ$-product preserves nonnegativity of signed symmetric factor sequences, section 3.5.1, a demonstration of the nonnegativity for ${}^{C}\underline{\mathcal{U}}$ suffices to demonstrate the nonnegativity of $\underline{\mathcal{U}}$. From section 3.4.4, the number of terms in the expansion (66) of $\mathcal{U}_{k,n-k}$ is finite, and the sequences $\underline{f} \in \underline{\mathcal{P}}$ are terminating.

From substitution of (55) into (54).

$${}^{C}\mathcal{U}_{n,m}(\{1, n+m\}) := \sum_{\pi_1} \sigma(\pi_1, (\kappa)_{n+m}) \left(\sum_{\pi_2} \sigma(\pi_2, (\kappa)_{n+m})\ \tilde{u}_{n,m}(\pi_2(\pi_1(\{1, n+m\}))) \right)$$

with

$$\begin{aligned}\tilde{u}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} &:= \iint d\sigma(\lambda)\frac{du}{(2\pi)^4}\,\lambda^{n+m}\prod_{j=1}^{n+m} e^{-ip_j u}\,\delta(p_j^2-\lambda_{cj}^{-2})\,\frac{d}{d\rho_j} \\ &\quad\times\exp(\sum_{a,b\in\mathbb{J}_{n,n+m}}\rho_a\rho_b H_{\kappa_a\kappa_b}(p_a,p_b))\end{aligned} \tag{74}$$

from (55). Then from (50), ${}^C\mathcal{U}(\underline{f}^*,\underline{f})\geq 0$ if $u(\underline{f}^*,\underline{f})\geq 0$ with

$$\begin{aligned}u(\underline{f}^*,\underline{f}) &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\,(D\cdot)_n\tilde{u}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} \\ &\quad\times\overline{\tilde{f}_n}(-p_n,\dots-p_1)_{\kappa_n\dots\kappa_1}\,\tilde{f}_m(p_{n+1},\dots p_{n+m})_{(\kappa)_{n+1,n+m}}\end{aligned} \tag{75}$$

using the scalar product (21). This scalar product is expressed using Fourier transforms (16) from Parseval's equality (17), and (8) for $*$-dual functions. The Fourier transform (16) of a $*$-dual function is

$$\int\frac{d(x)_n}{(2\pi)^{2n}}\,e^{-ip_1x_1}\dots e^{-ip_nx_n}f^*((x)_n)_{(\kappa)_n} = (D^T\cdot)_n\overline{\tilde{f}_n}((-p)_{n,1})_{(\kappa)_{n,1}}.$$

The factorization

$$\begin{aligned}(D\cdot)_n\exp(\sum_{a,b\in\mathbb{J}_{n,n+m}}\rho_a\rho_b H_{\kappa_a\kappa_b}) &= \exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}(DBD^T)_{\kappa_{a_1}\kappa_{b_1}}) \\ &\quad\times\exp(\sum_{b_2>a_2=n+1}^{n+m}\rho_{a_2}\rho_{b_2}B_{\kappa_{a_2}\kappa_{b_2}})\,\exp(\sum_{a_3=1}^{n}\sum_{b_3=n+1}^{n+m}\rho_{a_3}\rho_{b_3}(D\Upsilon)_{\kappa_{a_3}\kappa_{b_3}}).\end{aligned}$$

from (58) of section 3.4.3 results in display of $u(\underline{f}^*,\underline{f})\geq 0$ [37]. Three types of factors are distinguished. Substitution of the identity (40) into the definition (57) provides that

$$DB(p)D^T = B(-p)^\dagger. \tag{76}$$

The nonnegative matrix $DM(s)=C^\dagger(s)C(s)$ from (39) and then substitution into (57) provides that

$$D\Upsilon(p) = \int d\mu_\Upsilon(s)\,e^{-sp}C^\dagger(s)C(s). \tag{77}$$

The momentum dependence of the $H(p_1,p_2)_{\kappa_1\kappa_2}$ is in (56). The factor

$$\exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}(DBD^T)_{\kappa_{a_1}\kappa_{b_1}}) = \exp(\sum_{b_1>a_1=1}^{n}\rho_{a_1}\rho_{b_1}\overline{B(-p_{a_1}-p_{b_1})}_{\kappa_{b_1}\kappa_{a_1}})$$

after substitution of (76), the factor

$$\exp(\sum_{a=1}^{n}\sum_{b=n+1}^{n+m}\rho_a\rho_b(D\Upsilon)_{\kappa_a\kappa_b}) = \sum_{N=0}^{\infty}\frac{1}{N!}\left(\sum_{a=1}^{n}\sum_{b=n+1}^{n+m}\rho_a\rho_b(D\Upsilon(-p_a+p_b))_{\kappa_a\kappa_b}\right)^N$$

and the momentum dependence of the third factor is $B(p_{a_2}+p_{b_2})$. Then, linearity of summation provides

$$\sum_a\sum_b x_a y_b = \left(\sum_a x_a\right)\left(\sum_b y_b\right) \tag{78}$$

and reordering finite summations with the integration in (77) results in

$$\begin{aligned}\sum_{a=1}^{n}\sum_{b=n+1}^{n+m}\rho_a\rho_b(D\Upsilon(-p_a+p_b)))_{\kappa_a\kappa_b} &= \sum_{\ell=1}^{N_c}\int d\mu_\Upsilon(s)\left(\sum_{a=1}^{n}\rho_a e^{sp_a}\overline{C(s)}_{\ell\kappa_a}\right)\\ &\quad\times\left(\sum_{b=n+1}^{n+m}\rho_b e^{-sp_a}C(s)_{\ell\kappa_b}\right)\end{aligned}$$

using the definition of Hermitian transpose $C(s)^\dagger$. Then, in the second factor

$$\begin{aligned}\left(\sum_{a=1}^{n}\sum_{b=n+1}^{n+m}\rho_a\rho_b(D\Upsilon)_{\kappa_a\kappa_b}\right)^N &= \sum_{\ell_1=1}^{N_c}\int d\mu_\Upsilon(s_1)\ldots\sum_{\ell_N=1}^{N_c}\int d\mu_\Upsilon(s_N)\\ &\quad\times\prod_{\nu=1}^{N}\left(\sum_{a_\nu=1}^{n}\rho_{a_\nu}e^{s_\nu p_{a_\nu}}\overline{C(s_\nu)}_{\ell_\nu\kappa_{a_\nu}}\right)\left(\sum_{b_\nu=n+1}^{n+m}\rho_{b_\nu}e^{-s_\nu p_{b_\nu}}C(s_\nu)_{\ell_\nu\kappa_{b_\nu}}\right).\end{aligned}$$

With these substitutions into (74), $u(\underline{f}^*,\underline{f})$ is displayed as a linear combination with positive weights of manifestly nonnegative terms. First, define

$$\begin{aligned}F_N(\lambda,u,(s,\ell)_N) &:= \sum_n\sum_{(\kappa)_n}\lambda^n\int d(p)_n\;\tilde{f}_n((p)_n)_{(\kappa)_{1,n}}\prod_{j=1}^{n}\left(e^{-ip_j u}\,\delta(p_j^2-\lambda_{cj}^{-2})\,\frac{d}{d\rho_j}\right)\\ &\quad\times\exp(\sum_{b_2>a_2=1}^{n}\rho_{a_2}\rho_{b_2}B(p_{a_2}+p_{b_2})_{\kappa_{a_2}\kappa_{b_2}})\prod_{\nu=1}^{N}\left(\sum_{a_\nu=1}^{n}\rho_{a_\nu}e^{-s_\nu p_{a_\nu}}C(s_\nu)_{\ell_\nu\kappa_{a_\nu}}\right).\end{aligned}$$

This is evaluated at $(\rho)_n = 0$ after differentiation. Then, with the order of the arguments of the $*$-dual of $\tilde{f}_n$ returned to the order

$$p_1,\kappa_1,p_2,\kappa_2,\ldots p_n,\kappa_n$$

from $p_n, \kappa_n, p_{n-1}, \kappa_{n-1}, \ldots p_1, \kappa_1$ and with momenta reflected $p_j \mapsto -p_j$ for $j \in \{1, n\}$ by relabeling summation variables, (75) is

$$u(\underline{f}^*, \underline{f}) = \sum_{N=0}^{\infty} \frac{1}{N!} \iint d\sigma(\lambda) \frac{du}{(2\pi)^4} \sum_{\ell_1=1}^{N_c} \int d\mu_\Upsilon(s_1) \ldots \sum_{\ell_N=1}^{N_c} \int d\mu_\Upsilon(s_N) \; |F_N(\lambda, u, (s, \ell)_N)|^2$$

The n and the p_j, κ_j, ρ_j within the two factors of $F_N(\lambda, u, (s, \ell)_N)$ are independent variables, labeled $n, (p, \kappa)_n$ and $m, (p, \kappa)_{n+1,n+m}$ in (75).

As a consequence of the factoring of the $n \geq 4$ argument connected functions (58), $u(\underline{f}^*, \underline{f})$ is expressed as a linear combination with positve coefficients of complex conjugate factors. The measures du, $d\sigma(\lambda)$ and $d\mu_\Upsilon(s)$ are nonnegative. Then $u(\underline{f}^*, \underline{f})$ is manifestly nonnegative. (28) in section 3.2 provides that positive definiteness is preserved with implementation of elemental stability A.7.

The ${}^C\underline{\mathcal{U}}$ satisfy A.1-7 and consequently define a realization of relativistic quantum physics. These realizations are unphysical in the sense that free particles are not described by the states: if the supports of the spacetime arguments of f_n are isolated and widely space-like separated, then the connectivity of the ${}^C\underline{\mathcal{U}}$ implies that the norm of these states approach zero: only states with effectively overlapping supports have significant norms.

Finally, the ${}^C\mathcal{U}(\underline{f}^*, \underline{f})$ from (54) and (55) is nonnegative for function sequences $\underline{\mathcal{P}}$ and consequently $\underline{\mathcal{U}}$ provides a degenerate scalar product (21). $\underline{\mathcal{U}}$ is nonnegative and signed symmetric as a consequence of the expansion (66) and the nonnegativity and signed symmetry preserving properties of the $\circ$-product. Then, the signed symmetry and nonnegativity of $\underline{\mathcal{F}}$, section 3.3.1 and appendix 6.9, provide that the construction $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$ is signed symmetric and nonnegative. The constructed $\underline{\mathcal{W}}$ satisfy the positive definiteness condition A.2.

3.5.3 Poincaré invariance

The Poincaré covariance of the constructed VEV is established in [37]. Invariance of the degenerate scalar product (21) follows from the conservation of energy-momentum and properties (43) of the representations of the Lorentz subgroup. Compliant example representations of the Lorentz subgroup are included in section 3.3.2.

Poincaré invariance of likelihoods is that

$$\langle (a, \Lambda)\underline{g} | (a, \Lambda)\underline{f} \rangle = \langle \underline{g} | \underline{f} \rangle,$$

for Poincaré transformations (30),

$$(a, \Lambda)\tilde{f}_n((p)_n)_{(\kappa)_n} := \prod_{k=1}^{n} e^{-ipa} \, (S(A)^T \cdot)_n \tilde{f}_n((\Lambda^{-1}p)_n)_{(\kappa)_n}.$$

The scalar product (21), the Fourier transform (16), Parseval's equality (17), and the dual of functions (8) result in that

$$\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle = \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\ (D\cdot)_n \widetilde{\mathcal{W}}_{n,m}((p)_{n+m})_{(\kappa)_{n+m}} \prod_{\ell=1}^{n+m} e^{-ip_\ell a}$$
$$\times\overline{(S(A)^T\cdot)_n \tilde{g}_n((-\Lambda^{-1}p)_{n,1})}_{\kappa_{n,1}} (S(A)^T\cdot)_{n+1,n+m}\tilde{f}_m((\Lambda^{-1}p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}$$

in the matrix notation (9). Translation invariance is verified by noting that the support of $\widetilde{\mathcal{W}}_{k,n-k}$ includes only the surface with $p_1 + p_2 + \ldots p_n = 0$, the surface with energy-momentum conserved. For the constructed connected functions (46) in the cluster decomposition (60) with (67) and (68), each factor in the products of connected functions include

$$\delta(p_{i_1} + \ldots p_{i_\ell}) \ldots \delta(p_{i_\nu} + \ldots p_{i_n})$$

with each p_k appearing exactly once in the set of arguments. The consequent energy-momentum conservation, $p_1 + p_2 + \ldots p_n = 0$, implements translation invariance.

Reordering summations, substitution of $\overline{S(A)}D = DS(A)$ from (43), the indicated substitutions $\Lambda^{-1}p_j \mapsto p_j$ for summation variables, and that the determinant of the Lorentz transformation Λ is unity results in

$$\begin{aligned}\langle (a,\Lambda)\underline{g}|(a,\Lambda)\underline{f}\rangle &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(p)_{n+m}\ (D\cdot)_n (S(A)\cdot)_{n+m}\widetilde{\mathcal{W}}_{n,m}((\Lambda p)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad \times\overline{\tilde{g}_n}((-p)_{n,1})_{\kappa_{n,1}}\tilde{f}_m((p)_{n+1,n+m})_{(\kappa)_{n+1,n+m}} \\ &= \langle \underline{g}|\underline{f}\rangle\end{aligned}$$

if

$$(S(A)\cdot)_n\widetilde{\mathcal{W}}_{k,n-k}((p)_n) = \widetilde{\mathcal{W}}_{k,n-k}((\Lambda^{-1}p)_n).$$

There is exactly one factor of $S(A)$ for each field component argument κ_j and consequently two factors of $S(A)$ for every factor of $M(p_\ell)_{\kappa_j\kappa_\ell}$ or $H(p_j,p_\ell)_{\kappa_j\kappa_\ell}$ in each term in $\mathcal{W}_{k,n-k}$ from the composition (60) of connected functions and the constructions of sequences $\mathcal{F}$ and $\mathcal{U}$ from connected functions, (44) and (67) with (54) and (55), respectively.

By construction, the $N_c \times N_c$ matrices $M(p)$ in (37) and the $H(p_1,p_2)$ from (56) and (57) transform with the same representation of the Lorentz subgroup. From the condition (43) and that $((S(A)\cdot)_2 M(p)) = S(A)M(p)S(A)^T$ from (9),

$$S(A)M(p)S(A)^T = M(\Lambda^{-1}p)$$

and then if $H = B$,

$$\begin{aligned} S(A)H(p_1,p_2)S(A)^T &= \int d\mu_B(s)\; e^{is(p_1+p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_B(s)\; e^{is(p_1+p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_B(s')\; e^{is'\Lambda^{-1}(p_1+p_2)} M(s') \\ &= H(\Lambda^{-1}p_1, \Lambda^{-1}p_2) \end{aligned}$$

and if $H = \Upsilon$,

$$\begin{aligned} S(A)H(p_1,p_2)S(A)^T &= \int d\mu_\Upsilon(s)\; e^{s(p_1-p_2)} S(A)M(s)S(A)^T \\ &= \int d\mu_\Upsilon(s)\; e^{s(p_1-p_2)} M(\Lambda^{-1}s) \\ &= \int d\mu_\Upsilon(s')\; e^{s'\Lambda^{-1}(p_1-p_2)} M(s') \\ &= H(\Lambda^{-1}p_1, \Lambda^{-1}p_2) \end{aligned}$$

from the Lorentz invariance of the measures $d\mu_B(s)$ and $d\mu_\Upsilon(s)$, the substitution $s' = \Lambda^{-1}s$ for the summation variable, the Lorentz invariance of the Minkowski signature $\Lambda^T g \Lambda = g$, and that $ps := p^T g s$ is a Lorentz scalar. The generalized functions $\delta(p_{i_1} + \ldots p_{i_n})$ and $\delta(p_k^2 - \lambda_{ck}^{-2})$ are Lorentz scalars. These three cases, each κ_j uniquely associates with a factor of M, B or Υ, apply in every term of the expansion (60) with (44), (67), (54) and (55). Each factor of M or H introduces paired indices κ_a, κ_b with $a \neq b$, and as a consequence, with the exception of Lorentz scalar fields, only even order VEV appear in the constructions.

As a consequence of the conservation of energy-momentum, and representation of the Lorentz subgroup by $S(A)$ and Dirac conjugation D that satisfies (43), the constructed VEV provide a Poincaré invariant scalar product (21) that satisfies relativistic invariance A.3.

3.5.4 Cluster decomposition

Satisfaction of the strong form of cluster decomposition A.6 follows from the connectivity of the truncated functions (71) in the cluster expansion (70). This is an established result in RQFT [9] but the constructed $\underline{\mathcal{U}}$ do not generally satisfy formal Hermiticity W.a nor totality W.b. Satisfaction of cluster decomposition A.6 is demonstrated in this section. The demonstration is that for functions supported on distantly space-like separated volumes, scalar products factor.

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \to \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\, \mathcal{W}(\underline{g}^*, \underline{f}),$$

from (27) as the supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the supports of $\underline{g}, \underline{f}$.

Linearity of summations and relabeling the summation variables in the scalar product $\mathcal{W}(\underline{\psi}^*, \underline{\varphi})$ from (21) labeled by n, m and $(x,\kappa)_{n+m}$, and in $\mathcal{W}(\underline{g}^*, \underline{f})$ labeled by ℓ, k and $(y,\mu)_{\ell+k}$ results in

$$\begin{aligned}\mathcal{W}(\underline{\psi}^*, \underline{\varphi})\,\mathcal{W}(\underline{g}^*, \underline{f}) = \sum_{n.m}\sum_{\ell,k}\sum_{(\kappa)_{n+m}}\sum_{(\mu)_{\ell+k}}\iint d(x)_{n+m}d(y)_{\ell+k}\\ \times(D\cdot)_n\,\mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\,(D\cdot)_\ell\,\mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}}\\ \times\overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}}\varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}\overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}}f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{79}$$

The $\times$-product of function sequences (4), the $*$-dual of function sequences (8), and relabeling summation variables with the same designations for ψ, φ and g, f as (79) produces

$$\begin{aligned}\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) = \sum_{n.m}\sum_{\ell,k}\sum_{(\kappa)_{n+m}}\sum_{(\mu)_{\ell+k}}\iint d(x)_{n+m}d(y)_{\ell+k}\\ \times(D\cdot)_{n+\ell}\,\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})}\\ \times\overline{\psi}_n((x)_{n,1})_{(\kappa)_{n,1}}\varphi_m((x)_{n+1,n+m})_{(\kappa)_{n+1,n+m}}\overline{g}_\ell((y)_{\ell,1})_{(\mu)_{\ell,1}}f_k((y)_{\ell+1,\ell+k})_{(\mu)_{\ell+1,\ell+k}}.\end{aligned} \tag{80}$$

The assignment of summation variables is a partition of arguments into two subsets P with spacetime arguments $(x)_{n+m}$ and P' with spacetime arguments $(y)_{\ell+k}$ suitable to test satisfaction of cluster decomposition (27). To test the cluster decomposition (27), the supports of the ψ_n, φ_m are taken to be arbitrarily space-like separated from the support of the g_ℓ, f_k: every $x_j \in P$ can be considered as arbitrarily space-like separated from every $y_{j'} \in P'$.

In this notation, the construction (70) of $\underline{\mathcal{W}}$ has elements

$$\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} = (\exp\circ({}^C\mathcal{W}))_{n+\ell,m+k}$$

with the indicated association of arguments

$$(x,\kappa)_n,(y,\mu)_\ell,(x,\kappa)_{n+1,n+m},(y,\mu)_{\ell+1,\ell+k} \leftrightarrow (x,\kappa)_{1,n+m+\ell+k}$$

on the right- and left-hand sides, respectively. The number of the $*$-dual function arguments is designated n and m designates the number of the function arguments in $(x,\kappa)_{n+m}$. The remaining arguments are in $(y,\mu)_{\ell+k}$.

If

$$\begin{aligned}\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})}\\ = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\,\mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}},\end{aligned}$$

when the supports of every $x_j \in P$ is arbitrarily space-like separated from the support of every $y_{j'} \in P'$, then cluster decomposition is demonstrated.

The elements of the sequence ${}^C\underline{\mathcal{W}}$ from (46) are connected, and due to the great space-like separations of the $(x)_{n+m}$ from the $(y)_{\ell+k}$, any factor of ${}^C\mathcal{W}_{\nu_1,\nu_2}$ in any term of the expansion (66) for $(\exp\circ({}^C\mathcal{W}))$ is zero unless the arguments are all from $(x,\kappa)_{n+m}$ or are all from $(y,\mu)_{\ell+k}$. Then, designate

$$\delta_P((x)_{n+m},(y)_{\ell+k}) = \begin{cases} 1 & \text{if all arguments are } \in P \\ 0 & \text{otherwise} \end{cases} \tag{81}$$

and similarly for P'. Arguments P are arguments of ψ_n, φ_m and arguments P' are arguments of g_ℓ, f_k in the evaluation (80) of $\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f})$. Then, in the context of the scalar product (80),

$$\begin{aligned} \underline{\mathcal{W}} &= \exp\circ({}^C\underline{\mathcal{W}}) \\ &= \exp\circ(\delta_P{}^C\underline{\mathcal{W}} + \delta_{P'}{}^C\underline{\mathcal{W}}) \\ &= (\exp\circ(\delta_P{}^C\underline{\mathcal{W}}))\circ(\exp\circ(\delta_{P'}{}^C\underline{\mathcal{W}})) \\ &= (\delta_P\underline{\mathcal{W}})\circ(\delta_{P'}\underline{\mathcal{W}}) \end{aligned} \tag{82}$$

from the construction (70) and the identity (69). $\delta_P\underline{\mathcal{V}}$ designates the sequence (24)

$$(\mathcal{V}_{0.0}, \delta_P(x_1)\mathcal{V}_{1,0}(x_1)_{\kappa_1}, \ldots, \delta_P((x)_{n+m})\mathcal{V}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}, \ldots).$$

The factors of the elements of the sequence $\delta_P\underline{\mathcal{W}}$ have common spacetime arguments, and the designation of arguments as P or P' is established in the evaluation of (80). From (81), partitions that transpose P arguments with P' arguments do not contribute in the $\circ$-product (62) in the last line of (82). From $\mathcal{W}_{0,0} = 1$ and the identity (64) for the $\circ$-product (62), identification of the elements of the sequence $\underline{\mathcal{W}}$ in (82) produces

$$\begin{aligned} &\mathcal{W}_{n+\ell,m+k}((x)_n,(y)_\ell,(x)_{n+1,n+m},(y)_{\ell+1,\ell+k})_{((\kappa)_n,(\mu)_\ell,(\kappa)_{n+1,n+m},(\mu)_{\ell+1,\ell+k})} \\ &\qquad = \mathcal{W}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}}\,\mathcal{W}_{\ell,k}((y)_{\ell+k})_{(\mu)_{\ell+k}} \end{aligned}$$

in the context of the scalar product (80). Substitution into (80) and comparison with (79) completes the demonstration that for the construction (70) of $\underline{\mathcal{W}}$,

$$\mathcal{W}(\underline{\psi}^* \times \underline{g}^*, \underline{\varphi} \times \underline{f}) \to \mathcal{W}(\underline{\psi}^*, \underline{\varphi})\,\mathcal{W}(\underline{g}^*, \underline{f})$$

as the supports of $\underline{\psi}, \underline{\varphi}$ become arbitrarily distantly space-like separated from the supports of $\underline{g}, \underline{f}$.

The cluster expansion (70) for $\underline{\mathcal{W}}$ and the connectedness (46) of ${}^C\underline{\mathcal{W}}$ satisfy cluster decomposition A.6.

3.5.5 Regularity

Regularity A.1 states that the constructed VEV functions $\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are generalized functions from $\mathcal{S}'(\mathbb{R}^{4n})$. The construction, (60) with (65), (37), (44), (54) and (67), displays the VEV as finite sums of products of connected functions (46) with factors that have no arguments in common. Then, demonstration of regularity reduces to demonstration that the $n \geq 4$ argument connected functions are elements of $\mathcal{S}'(\mathbb{R}^{4n})$ since the two-point function (37) is an element of $\mathcal{S}'(\mathbb{R}^8)$. The $n \geq 4$ argument connected functions from (46), ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$, are products of elementary generalized functions but since products are not generally defined, these particular products must be justified.

To satisfy regularity A.1 for the physically nontrivial realizations, elemental stability A.7 is adopted. Elemental stability is implemented in (28) and the $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ in (55) describes interaction without introduction of a singular contributions from the extrapolation of ${}^C\mathcal{U}_{k,2n-k}$ to $2n = 2$ [33, 37]. For functions in $\underline{\mathcal{P}}$ and without vacuum polarization, conservation of energy sets all contributions from $\mathcal{W}_{0,n}$ to zero if $n \geq 2$. If kinematic constraints, for example, conservation of energy and angular momentum, preclude decay of isolated elementary particles, then satisfaction of formal Hermiticity W.a is consistent with $\mathcal{W}_{1,n-1}(f_1^* f_{n-1}) = 0$. More generally, setting $\mathcal{W}_{1,n-1}(f_1^* f_{n-1}) = 0$ is inconsistent with formal Hermiticity but generally implements regularity A.1 and positivite definiteness A.2 in the constructions.

Whether the connected functions ${}^C\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}$ are continuous linear functionals from $\mathcal{S}'(\mathbb{R}^{4n})$ rests on whether the summations over the indicated submanifolds of $(p)_n \in \mathbb{R}^{4n}$ in (46) are regular. The submanifolds are determined by Dirac delta functions in the connected functions (46). Each energy-momenta p_j lies on a mass shell and energy-momentum is conserved. If the singularities of $H(p_1, p_2)$ lie beyond the support of the Dirac delta functions, then their singularities are not a consideration. Contrasted with the neutral scalar field examples discussed in [33], multiple component fields introduce distinct elementary masses m_κ and $H(p_1, p_2)$ with singularities. There are no nontrivial finite Lorentz invariant measures [9] and as a consequence, $B(p)$ and $\Upsilon(p)$ from (56) with (57) are either constant or diverge at $p = 0$. From (57), nonconstant $B(p)$ are at least as singular as the Pauli-Jordan function [54]. From the singularities of the Pauli-Jordan function [9], and noting the Fourier transform property (18) and that the $M(p)$ in (56) are multinomials of the energy-momentum components, the $B(p)$ are as singular as

$$B(p) \approx \frac{d^{\mathbb{k}}}{dp^{\mathbb{k}}} \left(a\, \delta(p^2) + \frac{b}{p^2} \right) \tag{83}$$

in a neighborhood of $p^2 = 0$ and $\mathbb{k} > 0$ for nonzero spins. A constant $H(p_1, p_2)$ results from $d\mu_H(s) = c\delta(s)ds$ with $H = B, \Upsilon$. The energy-constrained support of functions from $\underline{\mathcal{P}}$ suffices to exclude the singularities of $H(p_1, p_2)$ from consideration in satisfaction of regularity A.1. The singularity of $B(p_j + p_k)$ is encountered only when p_j is the argument of a $*$-dual of a function ($E_j = -\hbar c\omega_j$) and p_k is the argument of a function ($E_k = \hbar c\omega_k$). In these instances, the points $(p_j + p_k)^2 = 0$ would be within the support of the summations that define the scalar product

(21). However, the selected form (55) with (56) includes only energy-momentum support p_j, p_k with either both $E_j, E_k > 0$ or both $E_j, E_k < 0$. Then $(p_j + p_k)^2 > m_{\kappa_j}^2 c^2 + m_{\kappa_k}^2 c^2$. Similarly for $\Upsilon(-p_j + p_k)$, the singularities are encountered only if both arguments are associated with a function or both are associated with the $*$-dual of a function. The form (56) is selected to exclude the singularities of $H(p_1, p_2)$ from the support of the summations that evaluate scalar products (21) [37].

In this section, the number of spacetime dimensions is considered and is designated d.

Each connected function (46) in the constructions is supported solely on submanifolds with energy and momentum conserved. After evaluation of the mass shell delta functions and with consideration of the zeros in the energy support of functions in $\underline{\mathcal{P}}$ and $\underline{\mathcal{P}}^*$, the Fourier transforms of the $n \geq 4$ connected functions include only momenta on the manifold defined by

$$\prod_{j=1}^{n} \frac{1}{2\omega_j} \, \delta(\omega_1 \ldots + \omega_k - \omega_{k+1} \ldots - \omega_n) \, \delta(\mathbf{p}_1 + \mathbf{p}_2 \ldots + \mathbf{p}_n) \tag{84}$$

with

$$\omega_j = \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}$$

from (13). Factors of $1/(2\omega_j)$ are multipliers of tempered functions for finite masses and are not considered further. Within the submanifold of $\mathbb{R}^{3n}$ with momentum conserved,

$$\mathbf{p}_n = -\mathbf{p}_1 \ldots - \mathbf{p}_{n-1}. \tag{85}$$

Energy conservation is $\delta(E_k((\mathbf{p})_n))$ with

$$E_k((\mathbf{p})_n) := \sum_{j=1}^{k} \omega_j - \sum_{j=k+1}^{n} \omega_j := \sum_{j=1}^{n} s_j \omega_j. \tag{86}$$

$\delta(E_k((\mathbf{p})_n))$ defines a generalized function except possibly for points on the surface $E_k((\mathbf{p})_n) = 0$ with a vanishing gradient, $\nabla E_k((\mathbf{p})_n) = 0$ [19].

$$s_j := \begin{cases} -1 & \text{if } j \in \{k+1, n\} \\ 1 & \text{otherwise.} \end{cases}$$

Next, the singularities of $\delta(E_k((\mathbf{p})_n))$ are identified.

The components of the gradient $\nabla E_k((\mathbf{p})_n)$ within the submanifold with momentum conserved are

$$\begin{aligned} \frac{dE_k((\mathbf{p})_n)}{d\mathbf{p}_j} &= s_j \frac{d\omega_j}{d\mathbf{p}_j} - \frac{d\omega_n}{d\mathbf{p}_n} \frac{d\mathbf{p}_n}{d\mathbf{p}_j} \\ &= s_j \frac{\mathbf{p}_j}{\omega_j} + \frac{\mathbf{p}_n}{\omega_n} \end{aligned} \tag{87}$$

from (86) with $d-1$ dimensional momentum vectors $\mathbf{p}_j$, the constrained $\mathbf{p}_n$ from (85), and $j \in \{1, n-1\}$. Summing squares provides that when the gradient vanishes, $\mathbf{p}_j^2/\omega_j^2 = \mathbf{p}_n^2/\omega_n^2$. Then, substitution of ω_j provides that the gradient vanishes if and only if

$$s_j \frac{\mathbf{p}_j}{m_{\kappa_j}} = -\frac{\mathbf{p}_n}{m_{\kappa_n}}$$

for each $j \in \{1, n-1\}$ and with the constrained $\mathbf{p}_n$ from (85). Then both the function and its gradient vanish,

$$E_k((\mathbf{p})_n) = \nabla E_k((\mathbf{p})_n) = 0,$$

only if

$$\sum_{j=1}^{n} s_j m_{\kappa_j} = 0 \tag{88}$$

from

$$\mathbf{p}_j = -s_j \frac{m_{\kappa_j}}{m_{\kappa_n}} \mathbf{p}_n$$

at the zeros. A neighborhood of those points with a zero energy and vanishing gradient is

$$\frac{\mathbf{p}_j}{m_{\kappa_j}} = s_j \frac{\mathbf{p}_1}{m_{\kappa_1}} + \mathbf{e}_j$$

with $\|\mathbf{e}_j\| < \epsilon \ll 1$, $0 < \|\mathbf{p}_1\|$ and $j \in \{2, n-1\}$. In this neighborhood,

$$\mathbf{p}_n = -\sum_{j=1}^{n-1} \mathbf{p}_j = -\frac{m_{\kappa_n}}{m_{\kappa_1}}\, \mathbf{p}_1 - \sum_{j=2}^{n-1} \mathbf{e}_j$$

using (88) and

$$\omega_j \approx \frac{m_{\kappa_j}}{m_{\kappa_1}} \omega_1 + s_j m_{\kappa_j} \frac{\mathbf{p}_1 \cdot \mathbf{e}_j}{\omega_1} + m_{\kappa_1} m_{\kappa_j} \frac{\mathbf{e}_j^2}{2\omega_1} - m_{\kappa_1} m_{\kappa_j} \frac{(\mathbf{p}_1 \cdot \mathbf{e}_j)^2}{2\omega_1^3}$$

to second order in small quantities and with $\mathbf{e}_n := -\sum_{j=2}^{n-1} \mathbf{e}_j$ and for $j \in \{2, n\}$.

On the submanifold with momentum and energy conserved, and within a neighborhood of the points where the gradient $\nabla E_k((\mathbf{p})_n)$ equals zero,

$$\begin{aligned} E_k((\mathbf{p})_n) &\approx \frac{m_{\kappa_1}}{2\omega_1^3} \sum_{j=2}^{n} s_j m_{\kappa_j} \left(\omega_1^2\, \mathbf{e}_j^2 - (\mathbf{p}_1 \cdot \mathbf{e}_j)^2 \right) \\ &:= \frac{R^2}{2\omega_1^3} \left(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2 \right) \end{aligned}$$

from (88), polar coordinates for the $(n-2)$ spatial vectors $\mathbf{e}_j$, and with the definitions

$$R^2 := \sum_{j=2}^{n-1} \mathbf{e}_j^2,$$

$\mathbf{u}_1 := \mathbf{p}_1/\|\mathbf{p}_1\|$ and

$$\alpha R^2 := m_{\kappa_1} \sum_{j=2}^{n} s_j m_{\kappa_j} \left(\mathbf{e}_j^2 - (\mathbf{u}_1 \cdot \mathbf{e}_j)^2\right), \qquad \beta R^2 := m_{\kappa_1} \sum_{j=2}^{n} s_j m_{\kappa_j}\, \mathbf{e}_j^2.$$

The result is that energy conservation (84) defines a generalized function except possibly for the points with the gradient $\nabla E_k((\mathbf{p})_n)$ vanishing when $E_k((\mathbf{p})_n) = 0$, the points with $R = 0$. A lack of coinciding singularities provides that

$$\begin{aligned} \delta(E_k((\mathbf{p})_n)) &= \delta(\frac{R^2}{2\omega_1^3}\,(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2)) \\ &= \frac{2\omega_1^3}{R^2}\,\delta(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2) + \frac{2\omega_1^3}{\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2}\,\delta(R^2). \end{aligned}$$

Since (84) is a generalized function for $R > 0$, $\delta(\alpha \mathbf{p}_1^2 + \beta m_{\kappa_1}^2)$ is regular.

For $n \geq 4$ and a sufficient number of dimensions d, the singularities of the energy-momentum conserving delta functions (84) are locally summable for the regular selection (56) of $H(p_1, p_2)$. The Jacobian for the polar coordinates for $(\mathbf{e})_{2,n-1}$ contributes $R^{(d-1)(n-2)-1}$ in d spacetime dimensions for the nth order connected functions. Then, the summations in evaluation of the degenerate scalar product (21) include

$$\frac{R^{(d-1)(n-2)-1}}{R^2}\,dR = R^{(d-1)(n-2)-3}\,dR$$

in the neighborhood of the singularities and $d \geq 3$ suffices for the summations to converge and the constructed $\underline{\mathcal{W}}$ to be continuous linear functionals dual to tempered functions. Terms

$$R^{(d-1)(n-2)-1}\delta(R^2)$$

do not contribute for $n \geq 4, d \geq 3$ [19]. $d \geq 3$ suffices for finite masses m_{κ_j} and $d \geq 4$ includes massless particles [36].

To include massless particles, $m_\kappa = 0$, the development follows the massive case except that new singularities are encountered. The positive and negative mass shells intersect at $\mathbf{p}^2 = 0$, and ω^{-1} and the derivatives of $\omega = \sqrt{\mathbf{p}^2}$ diverge at $\mathbf{p}^2 = 0$. Selection of basis function spaces with Fourier transforms with infinite order zeroes at each $\mathbf{p}_j^2 = 0$ regularizes the development

[36]. With massless particles, a neighborhood of the points with a vanishing gradient (87) is defined by

$$\frac{\mathbf{p}_j}{\sqrt{\mathbf{p}_j^2}} = s_j \frac{\mathbf{p}_1}{\sqrt{\mathbf{p}_1^2}} + \mathbf{e}_j$$

with the constraint

$$\mathbf{e}_j^2 + 2 s_j \frac{\mathbf{p}_1 \cdot \mathbf{e}_j}{\sqrt{\mathbf{p}_1^2}} = 0.$$

Unlike the massive particle case, the perturbations $\mathbf{e}_j$ must preserve the unit lengths of $\mathbf{p}_j/\sqrt{\mathbf{p}_j^2}$. The constraint to unit length reduces the number of degrees of freedom in the summation over $(\mathbf{e})_{2,n-1}$. In polar coordinates, the summation contributes $R^{(d-2)(n-2)-1}$ and as a consequence, $d = 4$ is required to satisfy regularity with the inclusion of massless particles.

Finite masses, the regular selection (56) for the arrays $H(p_1, p_2)$ and three or more spacetime dimensions d suffice for the constructed $\underline{\mathcal{W}}$ to satisfy regularity A.1. Inclusion of massless elementary particles requires four or more dimensions [36].

3.5.6 Summary of constructions

The construction of VEV functions $\underline{\mathcal{W}}$ from connected functions (46), (60) with (37), (44), (54) and (67), suffices to satisfy conditions A.1-7. The constructions are explicit example nontrivial realizations of relativistic quantum physics. The constructed VEV satisfy A.1-7 for states described by function sequences from $\mathbf{H}_{\mathcal{P}}$, the completion in the Hilbert space norm (20) of the basis function spaces $\underline{\mathcal{P}}$ (89). Or equivalently, the construction is for function sequences from $\underline{\mathcal{S}}$ with modified VEV, section 3.7.2. For the constructed VEV:

A.1: Demonstrations of regularity apply if the factors $Q_{k,2n-k}((p)_{2n})_{(\kappa)_{2n}}$ in the connected functions (46) are polynomially bounded growth, locally Lebesgue-summable functions of the energy-momenta over the appropriate domains (36), masses are finite, and the number of spacetime dimensions equals or exceeds three (2+1). Massless particles require four (3+1) or more spacetime dimensions and additional constraints on $\underline{\mathcal{P}}$, [36] and section 3.5.5. Regularity requires that the singularities of nonconstant $H(p_1, p_2)$ in the VEV (67) with (54) and (55) are excluded from the support of the scalar product (21), and that the divergent extrapolation of the connected functions (54) to two-point function is eliminated (28). A demonstration of regularity for finite masses is in section 3.5.5.

A.2: The nonnegativity of the scalar product (21) determined by the composition (60), $\underline{\mathcal{W}} = \underline{\mathcal{F}} \circ \underline{\mathcal{U}}$, follows from the nonnegativity of the signed symmetric (48) constituent sequences of generalized functions $\underline{\mathcal{F}}$ and $\underline{\mathcal{U}}$, section 3.5.1. The positive definiteness of $\underline{\mathcal{F}}$ is well established and is demonstrated in appendix 6.9. The nonnegativity of $\underline{\mathcal{U}}$ is demonstrated in section 3.5.2 and follows from the nonnegativity of ${}^{C}\underline{\mathcal{U}}$, the cluster decomposition (67), and that the $\circ$-product preserves the nonnegativity of signed symmetric sequences.

A.3: Section 3.5.3 includes a demonstration of relativistic invariance (29). Relativistic invariance follows from the relativistic covariance of $\underline{\mathcal{F}}$ and the expansions (54), (55) and (67) for $\underline{\mathcal{U}}$. The $S(A)$, $M(p)$ and D from the underlying free field $\underline{\mathcal{F}}$ establishes the realization of the Lorentz subgroup for a construction. Translation invariance follows from energy-momentum conservation.

A.4: The limitation of the support of VEV to mass shells together with the zeros (11) on negative energy mass shells in the support of the functions from the basis spaces $\underline{\mathcal{P}}$ implements the spectral support condition. In the evaluation of scalar products (21), either

$$p_j + p_{j+1} \ldots + p_n \in \overline{V}^+$$

explicitly since each $p_\ell \in \overline{V}^+$, or energy-momentum conservation provides that this sum of energy-momenta is the negative of $p_1 + p_2 \ldots + p_{j-1}$ with each of $p_1, \ldots p_{j-1}$ explicitly in the closed backward cone. The cones are closed under addition of elements. Demonstrations of spectral support are included in [33, 37].

A.5: For constructions based on function sequences $\underline{\mathcal{P}}$ from section 3.7, local commutativity follows from the signed symmetry of the constructed VEV (60). The $\circ$-product provides a signed symmetric (48) product of signed symmetric sequences.

A.6: Cluster decomposition follows from the connectedness of the functions ${}^C\underline{\mathcal{W}}$ in (46) and the cluster expansion (70) for the VEV functions $\underline{\mathcal{W}}$. The demonstration of cluster decomposition is in section 3.5.4.

A.7: VEV function sequences that satisfy A.1-6 determine sequences $\underline{\mathcal{W}}$ that also satisfy elemental stability A.7, (28). Elemental stability follows for mean zero fields if $\mathcal{W}_{k,0} = \mathcal{W}_{0,k} = \mathcal{W}_{k,1} = \mathcal{W}_{1,k} = 0$ for $k \geq 2$. For mean zero fields, $\mathcal{W}_{1,0} = \mathcal{W}_{0,1} = 0$. If kinematic constraints do not suffice to satisfy elemental stability, then the constructions must violate formal Hermiticity W.a. The introduction of vacuum polarizations preserves satisfaction of A.1-7 with the appropriately restated A.7, section 3.4.5.

For the example constructions, avoidance of the singularities of $H(p_1, p_2)$ to implement regularity (56) generally precludes formal Hermiticity W.a for the physically nontrivial constructions. The revised quantum-classical correspondence does not require Hermitian fields in the VEV (5). Free fields, Wick polynomials of free fields and generalized free fields satisfy formal Hermiticity W.a with a basis function space $\underline{\mathcal{S}}$. Satisfaction of A.1-6 with formal Hermiticity W.a and totality W.b appears to be peculiar to physically trivial VEV.

Generalizations to these constructions include odd order scalar field VEV [33], compositions that add connected functions $\underline{\mathcal{F}} \circ \underline{\mathcal{U}} \circ \underline{\mathcal{U}}'$, additional forms in the expansion (58) of connected functions in terms of the matrices $B(p)$ and $\Upsilon(p)$, extension of $d\mu_B(s)$ to complex-valued measures [33, 35], and massless particles, [36] and section 3.5.5.

3.6 Relation to free fields

The cluster expansion (65) and the connectedness of the VEV functions provide that the constructed realizations of RQP are described by free fields when the supports of each argument of the state describing functions are isolated (105). The connectedness and that at least two $*$-dual arguments and at least two function arguments appear in each contributing factor of ${}^{C}\mathcal{U}_{k,n-k}$ provides that the contributions of the constructed $\mathcal{W}_{k,n-k}$ reduce to contributions of $\mathcal{F}_{k,n-k}$ if the support of each argument of the state describing functions $\underline{f}$ are sufficiently widely space-like separated. In these instances, the states are readily interpreted as consisting entirely of (nearly) free particles: the scalar products of states with the support of each argument sufficiently widely space-like separated and localized nearly equal the scalar products for free fields. The two-point in (46) is a free field two-point function.

$$\langle \Phi(\underline{f})\Omega | \Phi(g)\Omega\rangle = \langle \Phi_o(\underline{f})\Omega_o | \Phi_o(g)\Omega_o\rangle_o$$

with $\Phi(\underline{f})$ the constructed quantum field (25) and $\Phi_o(\underline{f})$ the free field of the same elementary masses m_κ, $\langle \underline{f}|\underline{g}\rangle$ is the scalar product (21) using the constructed VEV and $\langle \underline{f}|\underline{g}\rangle_o$ is the free field, Fock space scalar product. These "macroscopic" state describing functions, with reliably distinguishable support, behave as particles with corresponding classical descriptions that satisfy Newtonian mechanics with $-g/r$ pair potentials in first approximation for nonrelativistic relative velocities and over brief intervals, section 4.

3.7 The basis function spaces

3.7.1 The basis function spaces $\underline{\mathcal{P}}$

A choice of basis function spaces $\underline{\mathcal{P}}$ admits nontrivial VEV realizations that are unavailable in Wightman's original development of relativistic quantum physics [56, 62]. Wightman selected the Schwartz tempered test functions $\underline{\mathcal{S}}$ and in this development, unrealizable constraints on VEV are relaxed by placing constraints on the function space. Wightman's selection of functions treats time similarly to space, and requires that the field operators that appear in the VEV (5) are Hermitian. In the constructions, time is distinguished: the functions in $\underline{\mathcal{P}}$ have Fourier transforms with zeros on negative energy mass shells (11).

$\underline{\mathcal{P}}$ consists of those tempered functions with Fourier transforms that vanish on the appropriate negative energy mass shells. The appropriate mass shell is determined for each field component labeled κ_j by the mass m_{κ_j}.

$$\varphi_n((x)_n)_{(\kappa)_n} \in \mathcal{P}(\mathbb{R}^{4n})$$

if

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = 0 \qquad \text{when any energy} \qquad E_j = -\hbar c\omega_j$$

for $j = 1, \ldots n$ and

$$\hbar c\omega_j := \hbar c\omega(\mathbf{p}_j) := \sqrt{(m_{\kappa_j}c^2)^2 + (\hbar\mathbf{p}_j)^2c^2}$$

from (13). The zeros are implemented using multiplier functions for tempered functions [20]. The functions $\varphi_n \in \mathcal{P}(\mathbb{R}^{4n})$ in (11) are constructed

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} := \prod_{j=1}^{n}(p_{j0} + \omega_j)\,\tilde{f}_n((p)_n)_{(\kappa)_n} \tag{89}$$

with $f_n \in \mathcal{S}(\mathbb{R}^{4n})$. This is abstracted as

$$\tilde{\mathcal{P}} := (p_0 + \omega)\tilde{\mathcal{S}}$$

using (89) and with $\varphi_0 = f_0 \in \mathbb{C}$. The factors $(p_{k0} + \omega_k)$ are multipliers of tempered test functions since they are infinitely differentiable and polynomially bounded in magnitude [20]. The remaining issue is whether A.3 is satisfied: whether $\underline{\mathcal{P}}$ is closed under Poincaré transformations (30). The Lorentz invariance of p^2 and that $(a, \Lambda)\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ for every $\tilde{g}_n \in \mathcal{S}(\mathbb{R}^{4n})$ provides that $\underline{\mathcal{P}}$ is stable under Lorentz transformations. Indeed, the Lorentz invariance of p^2 provides that a zero of the appropriate form (89) is stable with proper, orthochronous Lorentz transformation,

$$p'_{0,j} + \omega'_j = (0, \Lambda)(p_{j0} + \omega_j) = (p_{j0} + \omega_j)\frac{(p_{j0} - \omega_j)}{(0, \Lambda)(p_{j0} - \omega_j)}$$

for the Poincaré transformation (a, Λ). The final factor is regular in a neighborhood of the negative energy mass shell. The Poincaré transformation of functions is from (30).

$\mathbf{H}_\mathcal{P}$ includes states characterized by Cauchy sequences of functions in $\underline{\mathcal{P}}$ convergent in the Hilbert space norm (20). For the constructions, these include functions

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} = \prod_{j=1}^{n}(p_{j0} + \omega_j)\,\tilde{g}_n((\mathbf{p})_n)_{(\kappa)_n})$$

with $g_n \in \mathcal{S}(\mathbb{R}^{3n})$ and, as a consequence, $\mathbf{H}_\mathcal{P}$ includes states characterized by functions (227) with point support over time [33]. The functions used by Lehmann, Symanzik and Zimmermann to isolate the creation component of a field operator in their developments of scattering for RQFT [9] are included in $\mathbf{H}_\mathcal{P}$. As a consequence, LSZ expressions for scattering amplitudes readily adapt to the constructions. Explicit scattering amplitudes are presented in section 3.9 and [31, 33, 35, 37].

The Hilbert space completion of $\underline{\mathcal{P}}$ includes no strictly localized states [33] but there are essentially localized states. Functions of the form (89) are not of bounded spatial support. The essentially localized states include states arbitrarily dominantly supported within bounded

volumes, appendix 6.14, yet with tails that do not identically vanish within any finite volume. Functions of the form (89) are *anti-local* [45, 53]. The tails of the functions that describe states can be physically negligible since minor support corresponds to unlikely and effectively unrepeatable observations.

Selection of the basis function spaces $\underline{\mathcal{P}}$ overcomes the challenge of discovering VEV within $\underline{\mathcal{S}}'$ that satisfy the physical conditions of RQP for all functions from $\underline{\mathcal{S}}$. This long-standing problem remains without a resolution [2, 4, 9, 30, 39, 41, 56, 59]. The realizations of relativistic quantum physics discussed here are admitted by elimination of canonical quantization's assertion that quantum fields (25) are elevations of corresponding classical fields and, as a consequence, must be Hermitian. The lack of real functions within the completion of $\underline{\mathcal{P}}$ precludes Hermitian field operators but does not preclude an appropriate correspondence of classical and quantum dynamics.

3.7.2 An alternative formulation based on $\underline{\mathcal{S}}$

There is an equivalent formulation with VEV modified from the forms presented in sections 3.3.1 and 3.4 with the Schwartz tempered functions $\underline{\mathcal{S}}$ as the basis function space. In the equivalent formulation, the VEV are necessarily non-Hermitian when interaction is exhibited. The equivalent formulation includes the physically trivial, conventional free field VEV. These notes emphasize the development based on $\underline{\mathcal{P}}$. The two formulations are equivalent: basis spaces $\underline{\mathcal{P}}$ with the VEV displayed in (10) and section 3.4; or basis space $\underline{\mathcal{S}}$ with augmented VEV that are non-Hermitian if interaction is exhibited.

The equivalence derives from that the multiplier functions

$$\frac{\omega \pm p_0}{2\omega} = \theta(\pm p_0)$$

on mass shells $\hbar^2 p^2 = m^2 c^2$ with $\theta(x)$ the Heaviside step function. The zeros in (89) that distinguish the subspace $\underline{\mathcal{P}}$ from the tempered functions $\underline{\mathcal{S}}$ are provided by these multiplier functions that may alternatively be applied to the VEV. Application of the multipliers to the VEV produces the equivalent formulation with the basis function space as the Schwartz tempered functions $\underline{\mathcal{S}}$ as a consequence of the assignment (89). In this equivalent, alternative formulation, totality W.b is satisfied but formal Hermiticity W.a is abandoned to achieve interaction. With the multipliers applied to the VEV rather than to the test functions, the generalized functions become

$$\mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} \mapsto \prod_{j=1}^{k} \prod_{\ell=k+1}^{n} \left(\frac{-p_{j0} + \omega_j}{2\omega_j} \right) \left(\frac{p_{\ell 0} + \omega_\ell}{2\omega_\ell} \right) \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n}. \tag{90}$$

From section 3.4, the VEV have point support on energies on mass shells, $\delta(p_j^2 - m_{\kappa_j}^2 c^2/\hbar^2)$ for

each $j \in \{1, n\}$, and

$$\begin{aligned} p_{j0} + \omega_j &= 2\omega_j \theta(p_{j0}) \\ -p_{j0} + \omega_j &= 2\omega_j \theta(-p_{j0}). \end{aligned}$$

The resulting equivalence of formulations is:

$$\begin{array}{cc} \underline{\text{VEV}} & \underline{\text{Basis space}} \\ \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{P}} \\ \prod_{j=1}^{k} \theta(-p_{j0}) \prod_{\ell=k+1}^{n} \theta(p_{\ell 0})\, \mathcal{W}_{k,n-k}((x)_n)_{(\kappa)_n} & \underline{\mathcal{S}}. \end{array} \tag{91}$$

The equivalence of formulations is interesting for an analogy with a result from Reeh and Schlieder [45, 66]. The result of Reeh and Schlieder is that states described by (2) with Hermitian field operators, states described by operating on the vacuum with powers of field operators and functions $\varphi_n((x)_n)_{(\kappa)_n}$ supported solely within an open and bounded region χ, are dense in the entire Hilbert space and not just dense for states supported within the same bounded region χ. Although the theorem of Reeh and Schlieder is demonstrated for RQFT and not A.1-7, the equivalent formulations in (91) display the same puzzlement with states characterized by functions of bounded support but an inferred global support of state descriptions. The formulation with states labeled by the anti-local functions in $\underline{\mathcal{P}}$ is equivalent to a formulation that includes states labeled by functions of bounded support in $\underline{\mathcal{S}}$. Anti-local functions have global support in the sense that they do not vanish over any finite volume. The equality (91) demonstrates that the more evident anti-locality of the formulation based on $\underline{\mathcal{P}}$ is also exhibited in the formulations based on $\underline{\mathcal{S}}$: this suggests that anti-locality is implicit in relativistic quantum physics.

3.8 Properties of the constructed quantum fields

In this section, the Hamiltonains for the constructions are discussed and it is demonstrated that the fields $\Phi(\underline{f})$ constructed in section 3.4 are unbounded Hilbert space operators, and that the fields $\Phi(\underline{f})$ are not Hermitian if they are physically nontrivial.

As a consequence of the limitation of the momentum support of the VEV to mass shells and that the intersection of the supports of the VEV with the support (11) of functions from $\underline{\mathcal{P}}$ includes only positive energies, the Hamiltonian, the generator of time translations of a state describing function, derives from the single-argument subspace operator (32).

$$e^{-ip_{j0}\lambda} = e^{-i\omega_j \lambda} \tag{92}$$

using (13) and the translation is by $\lambda = ct$. This follows from (32) with the Fourier transform (16). In an n-argument subspace of $\mathbf{H}_{\mathcal{P}}$,

$$U(\lambda) = \prod_{j=1}^{n} e^{-i\omega_j \lambda} \tag{93}$$

implements temporal translation. The $\sum_{j=1}^{n} p_j$ in n-argument subspaces are densely defined Hermitian operators in the n-argument subspaces and correspond to total energy and momentum. Hermiticity follows from Poincaré invariance of the scalar product (21) and Stone's theorem [24]. In the subspace of the vacuum, $n = 0$, the Poincaré transformations (30) are $(a, \Lambda) = \mathbb{I}$. In multiple argument subspaces, an association of single argument subspace operators with the classical dynamical variables of corresponding particles is generally not determined. An association of the arguments of state describing functions with the properties of classical particles necessarily applies only for appropriate functions, section 4.1, or for VEV that lack interaction. The exhibition of interaction is described by the VEV (5): VEV determine the Hilbert space scalar product (6) and consequently the likelihoods of observations. The n and k-argument state describing functions are not orthogonal for $n \neq k$ with scalar products that exhibit interaction. The evolution of state describing functions and the exhibition of interaction are discussed further in sections 3.9 and 4.

Satisfaction of the axioms A.1-7 in section 3.2 suffices to define the constructed quantum fields $\Phi(\underline{f})$ as Hilbert space operators. The quantum fields (25) are the multiplication (4) of function sequences and this product preserves Hilbert space norm-equivalence classes. The expansion (3) for states as products of the field applies for function sequences from $\underline{\mathcal{P}}$: the domain of the field (25) includes this dense set of elements with $\mathbf{H}_{\mathcal{P}}$. Demonstrated below, the field is not a bounded operator and as a consequence, the field is at best only densely defined, [24], section 5 and appendix 6.2.5.

For a sufficiently great space-like translation T and an element $\underline{h}$ of the null space of $\mathbf{H}_{\mathcal{P}}$, satisfaction of cluster decomposition (27) in section 3.2 (axiom A.6) provides that

$$\begin{aligned}
\|\underline{f} \times (\underline{g} + T\underline{h})\|^2 &= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \times T\underline{h} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times T\underline{h} | \underline{f} \times T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | T\underline{h} \rangle + \langle T\underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle T\underline{h} | T\underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} \times \underline{g} | \underline{f} \rangle \langle \Omega | \underline{h} \rangle + \langle \underline{h} | \Omega \rangle \langle \underline{f} | \underline{f} \times \underline{g} \rangle + \langle \underline{f} | \underline{f} \rangle \langle \underline{h} | \underline{h} \rangle \\
&= \langle \underline{f} \times \underline{g} | \underline{f} \times \underline{g} \rangle \\
&= \|\underline{f} \times \underline{g}\|^2
\end{aligned}$$

from evaluation of the norm (20), linearity of the scalar product, translation invariance of the scalar product ($\|T\underline{h}\| = \|\underline{h}\|$), the Cauchy-Schwarz-Bunyakovsky inequality ($|\langle \underline{h} | \Omega \rangle| \leq \|\Omega\| \|\underline{h}\|$), and $\|\underline{h}\| = 0$. Elements of the null space are translationally invariant,

$$\|\underline{h} - T\underline{h}\| = 0,$$

a consequence of the Cauchy-Schwarz-Bunyakovsky inequality and the unitarity of translation $\|T\underline{h}\| = \|\underline{h}\|$. Then, the field operators $\Phi(\underline{f})$ in (25) preserve equivalence classes of the Hilbert space norm (20).

$$\|\Phi(\underline{f})\,(\underline{g}+\underline{h})\| = \|\Phi(\underline{f})\,\underline{g}\|$$

for any $\|\underline{h}\| = 0$. Repetition of this argument with $(\underline{f}+\underline{h})\times\underline{g}$ demonstrates that the quantum field $\Phi(\underline{f})$ is also independent of the representative used for the equivalence class of $\underline{f} \in \mathbf{H}_{\mathcal{P}}$. The constructed fields are Hilbert space operators but the fields are not Hermitian for the physically nontrivial constructions.

A quantum field $\Phi(\underline{f})$ (25) would be bounded if there is a real constant C such that $\|\Phi(\underline{f})\underline{v}\| \leq C\,\|\underline{v}\|$ for all $\underline{v} \in \mathbf{H}_{\mathcal{P}}$, [46] and appendix 6.2.5. However, the fields $\Phi(\underline{f})$ (25) are unbounded and consequently are not multiplier anywhere within $\mathbf{H}_{\mathcal{P}}$, [46] and section 5, and $\Phi(\underline{f})^*\underline{g} \notin \mathbf{H}_{\mathcal{P}}$ if $\underline{g} \in \mathbf{H}_{\mathcal{P}}$.

A demonstration of unboundedness for a free boson field suffices for the constructions of physical interest. Any construction that includes unconfined bosons includes the free boson field VEV. Then, the cluster decomposition axiom A.6 in section 3.2 provides that the field $\Phi(\underline{f})$ is unbounded if the free boson field is unbounded. For free bosons, the VEV functions result from

$$\mathcal{F}_{k,k}((x)_{2k})_{(\kappa)_{2k}} = \sum_{S}\prod_{j=1}^{k} W_2(x_j, x_{i_j})_{\kappa_j\kappa_{i_j}}$$

and are otherwise zero from (44) in section 3.3.1. For a product function,

$$\underline{f^n} := (0, 0\ldots, \prod_{\ell=1}^{n} f(x_\ell)_{\kappa_\ell}, \ldots),$$

application of (4) and (25) results in

$$\Phi(\underline{f})\underline{f^n} = (0, 0\ldots, f(x_1)_{\kappa_1}\prod_{\ell=2}^{n+1} f(x_\ell)_{\kappa_\ell}, \ldots).$$

Then the norm (20) for the free boson field provides

$$\|\underline{f^n}\|^2 = n!\; W_2(\underline{f}^*, \underline{f})^n$$

and

$$\|\Phi(\underline{f})\underline{f^n}\|^2 = (n+1)!\; W_2(\underline{f}^*, \underline{f})^{n+1}$$

with

$$W_2(\underline{f}^*, \underline{f}) := \sum_{\kappa_1=1}^{N_c}\sum_{\kappa_2=1}^{N_c} W_2(f^*_{\kappa_1}\, f_{\kappa_2})_{\kappa_1\kappa_2}$$

from (21). Then

$$\frac{\|\Phi(\underline{f})\underline{f^n}\|^2}{\|\underline{f^n}\|^2} = (n+1)\, W_2(\underline{f}^*, \underline{f}) > C^2$$

for any finite real constant C, some $n \in \mathbb{N}$ and some $\underline{f^n} \in \mathbf{H}_{\mathcal{P}}$. The constructed quantum fields $\Phi(\underline{f})$ are unbounded.

For VEV that satisfy formal Hermiticity W.a, the adjoints of the field (25) are readily evaluated. If the development also satisfies totality W.b, then the quantum fields are densely defined Hermitian Hilbert space operators. In Wightman's development or if formal Hermiticity W.a applies to a construction in this note, the definition of an adjoint results in the identification

$$\begin{aligned} \langle \Phi(\underline{f})^* \underline{h} | \underline{g} \rangle &= \langle \underline{h} | \Phi(\underline{f}) \underline{g} \rangle \\ &= \mathcal{W}(\underline{h}^*, \underline{f} \times \underline{g} \rangle \\ &= \underline{W}(\underline{h}^* \times \underline{f} \times \underline{g} \rangle \\ &= \underline{W}((\underline{f}^* \times \underline{h})^* \times \underline{g} \rangle \\ &= \langle \Phi(\underline{f}^*) \underline{h} | \underline{g} \rangle \end{aligned} \tag{94}$$

with $\underline{f} = (0, f(x_1)_{\kappa_1} \dots f(x_1)_{\kappa_{N_c}}, 0, 0, \dots)$ from section 3.1.3. This identification follows from the definitions of scalar product (21) and quantum field (25), the product of function sequences (4), and properties of the $*$-dual of sequences (8).

$$\mathcal{W}(\underline{f}^*, \underline{g}) = \underline{W}(\underline{f}^* \times \underline{g})$$

is formal Hermiticity W.a. Then, for VEV that satisfy formal Hermiticity W.a, the adjoint of the field is

$$\Phi(\underline{f})^* = \Phi(\underline{f}^*).$$

For VEV that satisfy formal Hermiticity W.a, the field is Hermitian, $\Phi(\underline{f})^* = \Phi(\underline{f})$, for real function sequences $\underline{f}^* = \underline{f}$. This defines a *real sequence* as $\underline{f}^* = \underline{f}$. In Wightman's original development [9, 33], totality W.b applies and the basis space of functions is the $*$-involutive $\underline{\mathcal{S}} = \underline{\mathcal{S}}^*$. With complex coefficients, real functions are dense in $\underline{\mathcal{S}}$: every $\underline{h} \in \underline{\mathcal{S}}$ decomposes as

$$\underline{h} = \underline{h}_1 + i\underline{h}_2$$

for real $\underline{h}_1, \underline{h}_2 \in \underline{\mathcal{S}}$, $2\underline{h}_1 = \underline{h} + \underline{h}^*$ and $2\underline{h}_2 = -i(\underline{h} - \underline{h}^*)$. For VEV and function sequences that satisfy Wightman's axioms, the field is densely defined and Hermitian. These VEV include the physically trivial free fields.

For constructions based on the support constrained function sequences $\underline{\mathcal{P}}$, any real function sequence $\underline{f}$ with $f_0 = 0$ is in the equivalence class of zero, [33] and section 3.7. For $\underline{f} \in \underline{\mathcal{P}}$ and VEV that satisfy formal Hermiticity W.a,

$$\Phi(\underline{f})^* \underline{g} = \underline{f}^* \times \underline{g} \notin \underline{\mathcal{P}}$$

unless $\underline{g} = 0$ since $\underline{\mathcal{P}} \cap \underline{\mathcal{P}}^* = \{(c, 0, 0 \ldots)\}$ with $c \in \mathbb{R}$. $\underline{\mathcal{P}}$ includes too few functions for the $*$-dual to be an automorphism. The constructed physically nontrivial fields are not Hermitian even if formal Hermiticity is satisfied due to the necessity of limited support for $\underline{\mathcal{P}}$ to implement nonnegativity of energies (11) and include higher order connected functions such as (10). The support constraints imply that the algebra of function sequences $\underline{\mathcal{P}}$ is not $*$-involutive and $\Phi(\underline{f})^* \underline{g} \notin \mathbf{H}_{\mathcal{P}}$ if $\underline{g} \in \mathbf{H}_{\mathcal{P}}$. The identification in section 3.3.1 and [31] of creation and annihilation operators for free fields illustrate this result. For $\underline{f} \in \mathcal{P}(\mathbb{R}^4)$ and free field VEV, the field operators $\Phi_o(\underline{f})$ in (25) equal the non-Hermitian creation operators $\Phi_o^+(\underline{f})$. The adjoints of $\Phi_o^+(\underline{f})$ are the annihilation operators $\Phi_o^-(\underline{f}^*)$ from (94). But, $\Phi_o^-(\underline{f}) = 0$ for $\underline{f} \in \mathcal{P}(\mathbb{R}^4)$ and this precludes Hermiticity of the field, $\Phi_o^+(\underline{f}) = \Phi_o(\underline{f}) \neq \Phi_o(\underline{f}^*) = \Phi_o^-(\underline{f}^*)$ for the energy support constrained functions $\underline{f} \in \mathcal{P}(\mathbb{R}^4)$ [31].

Generally, densely defined Hermitian adjoints of the quantum fields are precluded by axiom A.7. For physically nontrivial constructions, axiom A.7 is generally required to implement positive definiteness A.2 with regularity A.1: the divergent two-point function that results from extrapolation of (54) is eliminated using (28). If the quantum fields in (5) were Hermitian, then A.7 provides that

$$\begin{aligned} \langle \Phi(\underline{f}_n)^* \ldots \Phi(\underline{f}_2)^* \Phi(\underline{f}_1)^* \Omega | \underline{g} \rangle &= \langle \Phi(\underline{f}_n) \ldots \Phi(\underline{f}_2) \Phi(\underline{f}_1) \Omega | \underline{g} \rangle \\ &= \langle \Phi(\underline{f}_1)^* \Omega | \Phi(\underline{f}_2) \ldots \Phi(\underline{f}_n) \underline{g} \rangle \\ &= \langle \Phi(\underline{f}_1) \Omega | \Phi(\underline{f}_2) \ldots \Phi(\underline{f}_n) \underline{g} \rangle \\ &= \langle \Phi(\underline{f}_1) \Omega | \underline{f}_2 \times \ldots \underline{f}_n \times \underline{g} \rangle \\ &= 0 \end{aligned} \tag{95}$$

from the definition of adjoint operator and the assumed Hermiticity. Then, repeating the development with any number of factors, A.7 implies that any power of the quantum field vanishes if Hermitian field operators were realized. No nontrivial quantum field is Hermitian if A.1-7 are satisfied. A.7 is satisfied for free field VEV if the basis function sequences are limited to $\underline{\mathcal{P}}$: the free field VEV (44) extend to (38) with the extension of $\underline{\mathcal{P}}$ to $\underline{\mathcal{S}}$.[m]

Do realizations with VEV such as (10) exhibit interaction "at a point" or are they string theories? Of course, such geometric characterizations are not intrinsic to quantum mechanics but are motivated by a predisposition for classical description. Neither functions of point support nor functions of string support are included among the state describing functions in the constructions. The constructed Hamiltonians (92) generate temporal translations in the Hilbert space realizations of the Poincaré group. The Hamiltonian is determined by realization of the Poincaré group rather than a correspondence with the classical dynamics of points or strings. These Hamiltonians coincide with what is identified in the canonical formalism as free

[m] In RQFT, if $\Phi(\underline{f})\Omega = 0$ for a dense set of real functions $\underline{f} = \overline{\underline{f}}$, then $\Phi(\underline{f}) = 0$, (lemma 20.1 [9]). A.7 is expressed with the necessarily complex functions $\underline{f} \in \underline{\mathcal{P}}$.

field Hamiltonians even though interaction is manifest, sections 3.9 and 4, and [31, 33, 35]. The constructed VEV are solutions to the Klein-Gordon equation yet exhibit interaction. Mass shell singularities of the VEV together with a lack of constraints that set momenta equal in pairs result in nontrivial scattering. Canonical formalism-compliant solutions to the Klein-Gordon equation necessarily exhibit trivial physics. The Jost-Schroer theorem [9, 56] states that if quantum fields, the $\Phi(x)_\kappa$ in VEV (5), satisfy both the original Wightman axioms and the Klein-Gordon equation, then the fields $\Phi(x)_\kappa$ are free fields. The Jost-Schroer theorem does not apply with the revised axioms: interacting quantum fields are not necessarily densely defined Hermitian Hilbert space operators. The constructed scattering amplitudes that coincide with Feynman series at weak coupling demonstrates that solutions to the Klein-Gordon equation are of interest. For VEV that are solutions to the Klein-Gordon equation, interaction is inconsistent with Hermitian field operators. This suggests that interacting relativistic fields are precluded by canonical quantization's quantum-classical correspondence. The constructions suggest a strengthening of the Haag (Haag-Hall-Wightman-Greenberg) theorem to that Poincaré covariance, locality and positive energies together with an exhibition of interaction preclude Hermitian field operators.

> The pressure for conformity is enormous. I have experienced it in editors' rejection of submitted papers, based on venomous criticism of anonymous referees. The replacement of impartial reviewing by censorship will be the death of science. – Julian Schwinger.

3.9 Scattering amplitudes

Interaction is exhibited in changes to momenta and particle numbers. A lack of orthogonality of the descriptions of states with differing particle numbers and momenta implements interaction. From Born's rule, scalar products (21) provide the state transition likelihoods and to describe relativistic physics, these likelihoods must be Poincaré invariant. In the constructions, although the time translation of an n-argument function is an n-argument function, with the constructed scalar products, n-argument function do not necessarily correcpond to n elementary particles nor are the momenta necessarily equal in pairs. States with $k \neq n$-arguments become orthogonal to n-argument states when the supports of the state describing functions are widely space-like separated. Compton wavelengths (14) set the scale for wide separation. Connected contributions to VEV such as (10) implement interaction but their contributions become negligible as the supports of the arguments of state describing functions become widely space-like separated. Satisfaction of the cluster decomposition axiom A.6 provides that a scalar product is described by VEV approximated by free field VEV if the support of the state describing functions becomes widely space-like separated. Free fields have natural interpretations as classical particles [7, 9, 23, 24, 52, 61]. As initially localized and widely space-like separated supports propagate and approach overlap, exhibition of interaction manifests in the state

transition amplitudes: observations of distinct momenta and particle numbers become more likely.

Scattering amplitudes are proportional to large time difference transition amplitudes evaluated in a limit with incoming and outgoing states described as plane waves [7, 23, 52, 61]. For the example of a single neutral scalar field and states described by product functions $f_n((x)_n) = \prod_j \ell(x_j; \lambda_j, \mathbf{q}_j)$, the scattering amplitudes are

$$S_{n,m} := \lim_{\lambda\to\infty} \langle U(\lambda)\ell(\lambda, \mathbf{q}_{n+1}) \dots \ell(\lambda, \mathbf{q}_{n+m}) | U(-\lambda)\ell(-\lambda, \mathbf{q}_1) \dots \ell(-\lambda, \mathbf{q}_n)\rangle$$

with $U(\lambda)$ the unitary operator that translates the states in time. The parameters $\lambda_j, \mathbf{q}_j$ in the functions control an energy dependent phase and center the momentum support, respectively. The designation $\ell(\lambda_j, \mathbf{q}_j)$ indicates the values of the parameters of the state describing functions $\ell(x_j; \lambda_j, \mathbf{q}_j)$. The LSZ (Lehmann-Symanzik-Zimmermann) expressions [9] for scattering amplitudes use functions with Fourier transforms

$$\tilde{\ell}(p_j; \lambda_j, \mathbf{q}_j) := e^{iE_j\lambda_j}(\omega_j + E_j)\tilde{f}(\mathbf{p}_j - \mathbf{q}_j) \tag{96}$$

with λ_j a real parameter, $\mathbf{q}_j$ a momentum vector and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ is a Schwartz tempered test function. From section 3.7, $\ell(x_j; \lambda, \mathbf{q}_j)$ is a function in the completion $\mathbf{H}_\mathcal{P}$ of $\underline{\mathcal{P}}$. A convenient choice of test function are Gaussian functions

$$\tilde{f}(\mathbf{p}) = \left(\frac{L^2}{\pi}\right)^{3/2} e^{-L^2\mathbf{p}^2} > 0. \tag{97}$$

These $\tilde{f}(\mathbf{p})$ are point-wise nonnegative delta sequences heavily weighted near zero momentum when nearing the plane wave limit $L \to \infty$ and

$$\int d\mathbf{p}\ \tilde{f}(\mathbf{p} - \mathbf{q}) = 1.$$

The LSZ scattering amplitudes are VEV of products of fields

$$\Phi(\ell(\lambda, \mathbf{q})) := \int dp\ (\omega + E)e^{iE\lambda}\tilde{f}(\mathbf{p} - \mathbf{q})\ \tilde{\Phi}(p)$$

in this scalar field example. The VEV functions $\tilde{\mathcal{W}}_{k,n-k}((p)_n)$ are from (26) and section 3.4. Temporal translations of the field evaluated with the state describing functions (96) are independent of time with the selection of the state describing function parameter λ_j equal to λ.

$$\begin{aligned} U(\lambda)\Phi(\ell(\lambda, \mathbf{q}))U(\lambda)^{-1} &= \int dp\ (\omega + E)e^{-i(\omega-E)\lambda}\tilde{f}(\mathbf{p} - \mathbf{q})\ \tilde{\Phi}(p) \\ &= \Phi(\ell(0, \mathbf{q})) \end{aligned}$$

due to the limitation of the spectral support of the constructed VEV to mass shells.

In more familiar notation,

$$U(\lambda)\Phi(\ell(\lambda,\mathbf{q}))U(\lambda)^{-1} = i\int d\mathbf{x}\ \hat{u}(\lambda,\mathbf{x})\overset{\leftrightarrow}{\partial}_o \Phi(\lambda,\mathbf{x})$$

with

$$f(x)\overset{\leftrightarrow}{\partial}_o g(x) := f(x)\dot{g}(x) - \dot{f}(x)g(x),$$

$\dot{f}(x)$ the first time derivative of $f(x)$ and

$$\hat{u}(\lambda,\mathbf{x}) := \frac{1}{2\pi}\int d\mathbf{p}\ e^{i\omega\lambda}e^{-i\mathbf{p}\cdot\mathbf{x}}\tilde{f}(\mathbf{p}-\mathbf{q})$$

is a smooth solution of the Klein-Gordon equation.

For the Gaussian functions (97), the plane wave scattering amplitudes are the limits

$$\begin{aligned}\lim_{L\to\infty} S_{n,m} &= \lim_{\substack{L\to\infty\\ \lambda\to\infty}} \langle U(\lambda)\ell(\lambda,\mathbf{q}_{n+1})\dots\ell(\lambda,\mathbf{q}_{n+m})|U(-\lambda)\ell(-\lambda,\mathbf{q}_1)\dots\ell(-\lambda,\mathbf{q}_n)\rangle \\ &= \lim_{L\to\infty}\langle \ell(0,\mathbf{q}_{n+1})\dots\ell(0,\mathbf{q}_{n+m})|\ell(0,\mathbf{q}_1)\dots\ell(0,\mathbf{q}_n)\rangle.\end{aligned}$$

Evaluation of the mass shell deltas in the VEV from section 3 simplify the expression and the resulting quadrature is readily evaluated in the plane wave limit [33].

$$\begin{aligned}\lim_{L\to\infty} S_{n,m} &= \lim_{L\to\infty} c_{n+m}\left(\frac{L}{\sqrt{\pi}}\right)^{3(n+m)}\int d(\mathbf{p})_{n+m}\ 2^{n+m}\prod_{j=1}^{n+m}\omega_j e^{-L^2(\mathbf{p}_j-\mathbf{q}_j)^2} \\ &\qquad \times\widetilde{\mathcal{W}}_{n,m}((-\mathbf{p})_n,(\mathbf{p})_{n+1,n+m}) \qquad (98)\\ &= 2^{n+m}\prod_{j=1}^{n+m}\omega(\mathbf{q}_j)\widetilde{\mathcal{W}}_{n,m}((-\mathbf{q})_n,(\mathbf{q})_{n+1,n+m}).\end{aligned}$$

This introduced a convenient notation

$$\widetilde{\mathcal{W}}_{n,m}((-\mathbf{p})_n,(\mathbf{p})_{n+1,n+m}) \qquad (99)$$

for the VEV functions after evaluation of the mass shell delta functions, and $\omega(\mathbf{q}_j)$ is from (13). Each

$$\delta(p_k^2-\lambda_c^2)\mapsto\frac{\delta(E_k\pm\omega_k)}{2\omega_k}$$

and the sign is determined by whether p_k is the argument of a function or the $*$-dual of a function. Each energy is evaluated on the appropriate mass shell. The signs on the energies

$E_k = \pm\omega_k$ match the explicit signs of the momenta. Non-forward amplitudes result if only the connected contribution ${}^{C}\widetilde{\mathcal{W}}_{n,m}$ of the VEV is considered.

For the $n = 4$ example connected function (10), the non-forward, elastic scattering amplitude is

$$S_{2,2} = c_4\delta(\omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) - \omega(\mathbf{q}_3) - \omega(\mathbf{q}_4))\delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4). \tag{100}$$

This amplitude coincides up to a factor of i with the first order expansion from a Feynman series with an interaction Hamiltonian that includes a single neutral scalar field $:\Phi^4:$ term [31, 35]. This amplitude corresponds with a Yukawa-like equivalent potential in first Born approximation. The incoming momenta are $\mathbf{q}_1, \mathbf{q}_2$ and the outgoing momenta are $\mathbf{q}_3, \mathbf{q}_4$. The scattering amplitudes for (10) correspond to the first contributing order for an interaction Hamiltonian density $\mathcal{H}_{int}(x) = \sum_\ell a_\ell :\Phi(x)^\ell:$ with $\ell \geq 4$ and $a_\ell = c_\ell\,(2\pi)^{2\ell-4}/\ell!$. c_ℓ is from (53). The scattering cross sections of the constructions that realize quantum mechanics associate with first order (weak coupling) contributions from Feynman series. In cases with nonzero spin, for example, Compton scattering, the cross sections deviate from first order Feynman series results for extremely relativistic exchange momenta (small distances) [35].

In these example constructions, the scattering amplitudes are independent of time in the plane wave limit [33]. In this development, the propagation of the support of states can be followed through intermediate times. The lack of time dependence in finite time, plane wave limits is understood from a Hamiltonian that contributes only a phase in the plane wave limit and that plane waves uniformly cover all space.

4 Correspondence of quantum and classical descriptions

The discussion in section 2 illustrated that the presumed quantum-classical correspondence of canonical quantization imposes unrealizable constraints on realization of relativistic quantum physics. In this section, realizable correspondences of classical descriptions with the constructions from section 3 are developed further. A quantum-classical correspondence identifies classical descriptions that represent the support of those state describing functions with evident quantum-classical correspondences.

Understanding quantum mechanics as the description of nature has been a persistently controversial topic [5, 8, 16, 44, 51, 57, 65]. An example of discomfort with quantum mechanics is:

> We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. At least I do, because I'm an old enough man that I haven't got to the point that this stuff is obvious to me. Okay, I still get nervous with it... You know how it always is, every new idea, it takes a generation or two until it becomes obvious that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm not sure there's no real problem. – Richard Feynman, 1982, p. 471 in [17].

Quantum mechanics is often regarded as "strange," perhaps referring to the incompatibility of the quantum description of nature with classical understandings. Identified in the Einstein-Podolosky-Rosen (EPR) paradox, [16] and appendix 6.6, classical concepts are contradicted by the quantum description. The quantum description cannot even be thought of a statistical distribution over classical descriptions. In the classical concept, every particle is distinguishable and follows a trajectory. In the quantum description, particles are indistinguishable, and their location and momentum are never simultaneously known. Although the precision required to observe Heisenberg uncertainty bounds is unachievable on massive, "macroscopic" scales, Schrödinger's cat paradox [51] illustrates that quantum mechanics can not be relegated to a "strange" microscopic world.

Considerations for the quantum-classical correspondence include: whether classical descriptions correspond to the constructed state describing and VEV functions; when classical correspondences apply; and what VEV describe nature. The richness of quantum mechanics manifests in quantum-classical correspondences and a significant effort to characterize the correspondences remains.

Since the constructions realize quantum mechanics, finite interval transition amplitudes as well as infinite interval scattering amplitudes are available. Quantum-classical correspondences are richer than anticipated in RQFT: several distinct classical dynamical descriptions correspond to one construction. Scattering amplitudes and nonrelativistic brief duration amplitudes correspond to distinct classical dynamical descriptions. The finite interval, nonrelativistic state transition amplitudes are the primary focus of this section. For states that are well represented by finite mass, point-like particles with sufficiently slow relative velocities over sufficiently brief intervals, evolution is approximated by classical Newtonian mechanics with $-g/r$ pair potentials. Presumably, classical geometrodynamics and electrodynamics emerge from improved approximations.

The more widely studied quantum-classical correspondences are scattering amplitudes. Cluster decomposition A.6 provides that isolated initial and final scattered states are readily interpreted as free particles. Scattering amplitudes include the inherently relativistic massless particles. Scattering amplitudes enable comparison of the constructions with Feynman series. Scattering amplitudes are provided in [31, 33, 35] and section 3.9. From an RQFT perspective, the constructions are successful descriptions for nature even without studying the finite interval state transitions: the constructed scattering amplitudes include approximations to Feynman series.

The quantum-classical correspondence that is the primary topic of this section appears in Schrödinger's 1926 study of nonrelativistic linear harmonic oscillators [49]. The supports of selected solutions to the Schrödinger equation are well represented by corresponding classical dynamical variables. In 1+1 spacetime, particular Gaussian functions $\psi(x, \lambda)$ satisfy the

Schrödinger equation.

$$\psi(x, ct) = \exp\left(-\frac{(x - A\cos wt)^2}{4\sigma^2} - i\frac{\beta x \sin wt}{\hbar} - i\phi(t)\right)$$

satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\psi = i\hbar\frac{d\psi}{dt}$$

and the solution is characterized by a mass m, spring constant k, and oscillation amplitude A,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}}, \qquad w = \sqrt{\frac{k}{m}}$$
$$\beta = \sqrt{mk}\,A, \qquad \phi(t) = \frac{w}{2}t - \frac{kA^2}{4\hbar w}\sin 2wt.$$

The breadth of support of $\psi(x, ct)$ over x is described by σ and w is the oscillation frequency. Selected as the peak likelihood, the representative for the support of this function is the trajectory

$$x(\lambda) = A\cos w\lambda/c$$

of classical linear harmonic motion. The quantum-classical correspondence is most evident when the spread of support is small with respect to the amplitude of the motion, $\sigma^2 \ll A^2$. As $\sigma^2/A^2 \to 0$, the classical representative becomes essentially indistinguishable from the quantum description. As $m \to \infty$,

$$\sigma^2 = \frac{\hbar}{2\sqrt{mk}} \ll A^2 = \frac{2E}{k} \tag{101}$$

with the total classical energy $E = \frac{1}{2}kA^2$ a constant of the motion. The support of the function $\psi(x, \lambda)$:

1. persists along classical trajectories with the evolution of time λ:
2. the support of the state with the classical correspondence has a particular spread σ determined by k and E;
3. these $\psi(x, \lambda)$ are not eigenstates of the Hamiltonian, but are among the most-classical-like descriptions that meet the Heisenberg lower bound on the breadth of support over location and momentum;
4. and any classical energy E can be matched by a quantum mechanical description, a function $\psi(x, ct)$.

This quantum-classical correspondence does not apply for all state describing functions. For example, the supports of the energy eigenfunctions of a linear harmonic oscillator,

$$e_n(x, ct) = H_n(\alpha x) \exp\left(-i(n + \frac{1}{2})wt - \frac{1}{2}\alpha^2 x^2\right)$$

with H_n the nth Hermite polynomial [1] and

$$\alpha^2 = \frac{\sqrt{mk}}{\hbar},$$

do not follow classical trajectories of linear harmonic motion. The energy eigenfunctions exhibit quantized energies

$$\frac{1}{2}kA^2 = (n + \frac{1}{2})\hbar w.$$

For the energy eigenfunctions, only a phase, and not the support of the state describing functions, evolves with time.

To simplify notation and the development, discussions within this section are often limited to a single, neutral scalar field, $N_c = 1$.

4.1 States with a quantum-classical correspondence

Verified daily, classical dynamical variables provide accurate representatives for the quantum description when the support of states is "macroscopic," that is, classical body-like. A quantum state description is classical body-like if its spatial support is isolated and well represented by a single location, and the support of its Fourier transform is well represented by a single momentum. The Michelson interferometer illustrates that localization is necessary, and a vessel of gas illustrates that isolation is necessary to apply a quantum-classical correspondence, appendix 6.5. The EPR paradox [16] illustrates that descriptions must not be entangled for a quantum-classical correspondence to apply. If the dominant support of states is classical body-like, an approximate and conditional correspondence of classical and quantum state descriptions substitutes for the canonical quantization-conjectured elevations of classical dynamical variables to densely defined Hermitian operators.

Localization (102), isolation (105), a dominant momentum (103) and nonrelativistic support (109) are described below for one neutral scalar field and more generally, the definitions apply for each constituent function labeled by field component κ.

Localization: The jth argument of a state describing function $\varphi_n((x)_n)$ is localized near $\mathbf{y}_j$ if

$$\langle\psi_n|g(\mathbf{x}_j)\varphi_n\rangle \approx g(\mathbf{y}_j)\langle\psi_n|\varphi_n\rangle \tag{102}$$

for multiplier functions $g(\mathbf{x})$ of slow variation within the dominant support of $\varphi_n((x)_n)$ over $\mathbf{x}_j$. $\mathbf{y}_j$ is representative of the argument $\mathbf{x}_j$, the jth argument of $\varphi_n((x)_n)$. The notation (23)

is for function sequences with a single nonzero constituent function and $g(\mathbf{x}_j)\varphi_n$ designates the n-argument function

$$g(\mathbf{x}_j)\varphi_n(x_1, x_2, \ldots x_j, \ldots x_n).$$

Dominant momenta: The jth argument of a state description $\varphi_n((x)_n)$ is dominantly suppored near a momentum $\mathbf{q}_j$ if the support of the Fourier transform $\tilde{\varphi}_n((p)_n)$ is localized (102). That is,

$$\langle\tilde{\psi}_n|\tilde{g}(\mathbf{p}_j)\tilde{\varphi}_n\rangle \approx \tilde{g}(\mathbf{q}_j)\langle\tilde{\psi}_n|\tilde{\varphi}_n\rangle \tag{103}$$

for momentum domain, multiplier functions $\tilde{g}(\mathbf{p})$ of slow variation within the dominant support of argument $\mathbf{p}_j$ from $\tilde{\varphi}_n((p)_n)$. $\mathbf{q}_j$ is representative of $\mathbf{p}_j$, the jth argument of $\tilde{\varphi}_n((p)_n)$.

Isolated: The support of a localized argument $\mathbf{x}_j$ is isolated if there is little likelihood that a neighborhood of its representative value $\mathbf{y}_j$ is included within the dominant support of any other argument of the state describing function. To describe isolation, a localization of φ_n is developed. Every $\varphi_n \in \mathcal{P}(\mathbb{R}^{4n})$ has a Fourier transform of the form (89), section 3.7. In the notation of this section,

$$\tilde{\varphi}_n((p)_n) := \prod_{j=1}^{n}(p_{j0} + \omega_j)\, \tilde{f}_n((p)_n)$$

and then

$$\varphi_n((x)_n) = \prod_{j=1}^{n}\left(-i\frac{\partial}{\partial x_{j0}} + \sqrt{\lambda_c^{-2} - \Delta_j}\right) f_n((x)_n) \tag{104}$$

with $f_n((x)_n) \in \mathcal{S}(\mathbb{R}^{4n})$, Δ_j is the Laplacian for $\mathbf{R}^3$ that applies to the jth argument, and

$$\sqrt{\lambda_c^{-2} - \Delta_j}$$

is an anti-local operation, [45, 53] and appendix 6.14. A mapping $L_\ell(\chi, \mathbf{y}_j)\varphi_n$ of the function φ_n follows from

$$f_n((x)_n) \mapsto g_\chi(\mathbf{x}_\ell; \mathbf{y}_j) f_n((x)_n)$$

where $f_n((x)_n)$ is related to $\varphi_n((x)_n)$ by (104), and $g_\chi(\mathbf{x}; \mathbf{y}_j) \in \mathcal{S}(\mathbb{R}^{4n})$ is a multiplier function that equals one in a neighborhood $\chi \subset \mathbb{R}^3$ including $\mathbf{y}_j$ and is zero otherwise except for a brief, smooth transition from zero to unity [19]: the support of $g_\chi(\mathbf{x}; \mathbf{y}_j)$ is over a neighborhood χ of $\mathbf{y}_j$. The localization $L_\ell(\chi, \mathbf{y}_j)$ applies to the ℓth argument of φ_n. Then, the support of a localized argument $\mathbf{x}_j$ is isolated if

$$\|L_\ell(\chi, \mathbf{y}_j)\varphi_n\| \ll \|L_j(\chi, \mathbf{y}_j)\varphi_n\| \qquad \text{for} \quad \ell \neq j \tag{105}$$

and any neighborhood χ within the dominant support of argument j of φ_n. If isolation is satisfied, only the support of the jth argument associates to great likelihood with the volume of space represented by $\mathbf{y}_k$.

The most classical-like descriptions meet the Heisenberg uncertainty bound for localization and dominance in momentum. The association of spatial volumes with classical dynamical variables occurs on relatively large spatial scales, for spatial supports of extents large with respect to Compton wavelengths.

The quantum-classical correspondence discussed in this section applies only as long as the support of a state description remains isolated and well represented by a single location and momentum, and particle number is conserved. The nonrelativistic quantum-classical correspondences discussed below are limited to massive elementary particles, $m > 0$. Massless particles, inherently relativistic, are included in the constructions of section 3 but not in this discussion of quantum-classical correspondences.

Consistent with a nonrelativistic development, classical trajectories $\mathbf{u}_j(\lambda)$ are spatial vectors,

$$\mathbf{u}_j(\lambda) = (u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)),$$

defined in a particular reference frame. Associated spacetime vectors $u_j(\lambda)$ are designated

$$u_j(\lambda) = (0, u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)). \tag{106}$$

Derivatives with respect to the temporal parameter λ are designated

$$\dot{\mathbf{u}}_j(\lambda) := \frac{d\mathbf{u}_j(\lambda)}{d\lambda} \tag{107}$$

and the distance $\lambda := ct$ for a time t with c the speed of light. Trajectories are twice differentiable curves in $\mathbb{R}^3$ that specify one body's history of locations $\mathbf{u}_j(\lambda)$ and velocities $\dot{\mathbf{u}}_j(\lambda)$. Momenta $\hbar\mathbf{w}_j(\lambda)$ are Euclidean three-vectors associated with the particle trajectories $\mathbf{u}_j(\lambda)$.

$$\begin{aligned} \lambda_c \mathbf{w}_j(\lambda) &:= \gamma_j \, \dot{\mathbf{u}}_j(\lambda) \\ &\approx \dot{\mathbf{u}}_j(\lambda) \end{aligned} \tag{108}$$

with

$$\gamma_j := \frac{1}{\sqrt{1 - \dot{\mathbf{u}}_j^2}}$$

and the approximation applies in nonrelativistic ($\dot{\mathbf{u}}_j^2 \ll 1$) instances. λ_c is the reduced Compton wavelength (14) for the appropriate mass determined from the field component. The energy-momentum Lorentz vector is $p_j := (\omega(\mathbf{w}_j), \mathbf{w}_j)$ with $\omega(\mathbf{p})$ from (13).

Nonrelativistic: Nonrelativistic physics applies if the momentum domain supports of the functions φ_n are sufficiently limited. The support of an argument $\mathbf{p}_j$ is nonrelativistic if there is a boost to a reference frame such that the momentum support satisfies $\hbar^2 \mathbf{p}_j^2 \ll (mc)^2$ for $\mathbf{p}_j$

within the dominant support. In this frame,

$$\begin{aligned} \mathbf{p}_j^2 &\ll \lambda_c^{-2} \\ \langle\varphi_n|\omega_j\varphi_n\rangle &\approx \langle\tilde{\varphi}_n|(\lambda_c^{-1}+\tfrac{1}{2}\lambda_c\mathbf{p}_j^2)\tilde{\varphi}_n\rangle \\ \mathbf{w}_j(\lambda) &\approx \lambda_c^{-1}\dot{\mathbf{u}}_j(\lambda) \end{aligned} \tag{109}$$

from (106). Nonrelativistic approximations for the Hamiltonian (92) are discussed in appendix 6.13. A state description φ_n is *nonrelativistic* if there is a reference frame with the support of all n arguments nonrelativistic.

If the support of every argument of two functions f_n, g_m is nonrelativistic in a selected reference frame, then

$$\langle\varphi_n|\psi_m\rangle = 0 \quad \text{if } n \neq m.$$

In nonrelativistic instances, energy is not conserved if $n \neq m$ since from (109), $\omega_j \approx \lambda_c^{-1}+\frac{1}{2}\lambda_c\mathbf{p}_j^2$ and $\lambda_c^{-1} \gg \lambda_c\mathbf{p}_j^2$ for all constituent j. Then there is no solution with conserved energy, $k\lambda_c^{-1} = n\lambda_c^{-1}$, except for $n = k$. In nonrelativistic instances or if localized supports are sufficiently isolated, the scalar product (21) simplifies.

$$\begin{aligned} \langle\underline{\varphi}|\underline{\psi}\rangle &= \sum_{n,m}\langle\varphi_n|\psi_m\rangle \\ &\approx \sum_{n}\langle\varphi_n|\psi_n\rangle. \end{aligned}$$

Appropriate functions: If $\tilde{f}_n((\mathbf{p})_n, \lambda) \in \mathcal{S}(\mathbb{R}^{3n})$ is a function dominantly supported near $(\mathbf{p})_n = 0$ with inverse Fourier transforms dominantly supported near $(\mathbf{x})_n = 0$, then state describing functions

$$\tilde{\varphi}_n((p)_n;\lambda) = \prod_{j=1}^{n} e^{-i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0}+\omega_j)\tilde{f}_n((\mathbf{p}-\mathbf{w}(\lambda))_n;\lambda) \tag{110}$$

have spatial supports centered on $\mathbf{u}_j(\lambda)$ and momentum supports centered on $\mathbf{w}_j(\lambda)$. If the support of $\tilde{f}_n((\mathbf{p})_n, \lambda)$ nearly satisfies the Heisenberg uncertainty bound for location and momentum support breadth, then the state describing functions (110) are appropriate for a quantum-classical correspondence. Remaining requirements are that the description lacks entanglement and the $\mathbf{u}_j(\lambda)$ are sufficiently space-like separated to satisfy isolation. With appropriate selections for $\mathbf{u}_j(\lambda)$ and $\mathbf{w}_j(\lambda)$, the support of (110) is nonrelativistic (109), isolated (105) and localized (102). ω_j is from (13). From the Fourier transform pair (33) with (34), (111) has an inverse Fourier transform (16),

$$\varphi_n((x)_n;\lambda) := \prod_{j=1}^{n} e^{i\mathbf{w}_j(\lambda)\cdot(\mathbf{x}_j-\mathbf{u}_j(\lambda))}\left(-i\frac{\partial}{\partial x_{j0}}+\sqrt{\lambda_c^{-2}-\Delta_j}\right) f_n((x-u(\lambda))_n;\lambda) \tag{111}$$

with Δ_j the Laplacian in $\mathbb{R}^3$ for argument $\mathbf{x}_j$. The inverse Fourier transforms (16) of (110) have point support at time zero, each $x_{j0} = 0$, (227) in appendix 6.10. In (111), classical trajectories $(0, \mathbf{u}_j(\lambda))$ are representatives for the volumes of dominant spatial support, and the $\hbar \mathbf{w}_j(\lambda)$ are representatives for the dominant momentum support. The temporal parameter λ is included in the description of f_n in (111) to describe a continuous deformation of $f_n((x)_n; \lambda)$ with time. The example below is a λ dependent support spread parameter for a Gaussian function. The support of the state describing function translates and typically spreads with time evolution (32). The functions $\varphi_n((\mathbf{x})_n; \lambda)$ are infinitely differentiable with respect to $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$.

Products of single argument state describing functions lack entanglement and independently describe each argument: single particle observables do not vary with the descriptions of other particles. Products of Gaussian functions (97) are classical-like since products lack entanglement and Gaussian functions include the most classical-like single particle descriptions. From the Heisenberg uncertainty principle, appendix 6.7, Gaussian functions (97) with real spread parameters achieve the lower bound for support spread in location and momentum. Over intervals of limited duration, Gaussian functions translate in time to Gaussian functions with complex spread parameters in nonrelativistic approximation.

4.2 The support of state descriptions

The physically relevant support of states is determined by the likelihoods of observation. Likelihoods are determined from Born's rule from the squared magnitudes of scalar products (21) for state describing functions $\varphi_n \in \mathbf{H}_\mathcal{P}$. From A.3, Born's rule likelihoods are invariant to Poincaré transformations and state descriptions are covariant. The spatial support of $\varphi_n((x)_n)$ follows from the real-valued function over $(\mathbf{y})_k$,

$$|\langle (\mathbf{y})_k | \varphi_n \rangle|^2 \tag{112}$$

with $|(\mathbf{y})_k\rangle$ designating a state describing k-argument function in a delta sequence within $\mathbf{H}_\mathcal{P}$. $|(\mathbf{y})_k\rangle$ is centered on $(\mathbf{y})_k \in \mathbb{R}^{3k}$. In nonrelativistic instances, $k = n$. However, due to the very localized support of the delta sequence, the momentum domain support of a delta sequence includes very relativistic momenta. With the scalar products constructed in section 3, a state describing function $\varphi_n((x)_n)$ include descriptions of $k \neq n$ particles. Time translations of a spatial delta function will not be of point support; the Hamiltonian (93) is an anti-local operator. Similarly, the momentum support of $\varphi_n((x)_n)$ derives from

$$|\langle (\mathbf{q})_k | \tilde{\varphi}_n \rangle|^2 \tag{113}$$

with $(\mathbf{q})_k$ designating a k-argument function in a delta sequence within the Fourier transform domain of $\mathbf{H}_\mathcal{P}$ centered on $(\mathbf{q})_k$. The spatial support of $(\mathbf{q}_k)$ approaches $\mathbb{R}^3$ but $k = n$ for nonrelativistic momenta.

The energy-momentum support of the VEV constructed in section 3 is limited to mass shells. As a consequence, the time and spatial support of functions $\varphi_n((x)_n)_{(\kappa)_n}$ estimated as

the inverse Fourier transform of $\tilde{\varphi}_n((p)_n)_{(\kappa)_n}$ is ambiguous. Energy and momentum variables can be freely substituted

$$p_{j0} \longleftrightarrow \sqrt{\lambda_{cj}^{-2} + \mathbf{p}_j^2}.$$

The inverse Fourier transform of a momentum domain state describing function depends on whether one considers any p_{j0} an independent energy variable or a function of the momentum. In this section, a convention for the spacetime support of φ_n estimated as the inverse Fourier transform of $\tilde{\varphi}_n((p)_n)_{(\kappa)_n}$ is selected. The support over time of state describing functions is selected as point support: states are observed at an assigned time. Then, the convention is that there is no energy dependence of functions $\varphi_n \in \mathbf{H}_{\mathcal{P}}$, each $p_{j0} \mapsto \omega_j$.

In a selected reference frame, a state is described by a function over space and time parameterizes the evolution of that state description. Time generally is kept externally to the observed and perceived quantities are assigned a time. The completion of $\mathcal{P}(\mathbb{R}^{4n})$ to $\mathbf{H}_{\mathcal{P}}$ includes functions such as (111) that are generalized functions of point support over time and test functions over space. These are the functions of interest in nonrelativistic physics. Relativity covariantly relates descriptions among distinct reference frames and the likelihoods of events is independent of inertial reference frame.

From (6), (8), (16), (26) and the Fourier transform of generalized functions [19], the scalar product (21) expressed in the momentum domain is (114).

$$\begin{aligned}
\langle \underline{\varphi} | \underline{\psi} \rangle &= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(p)_{n+m} \, \langle \tilde{\Phi}(p_1)_{\kappa_1} \dots \tilde{\Phi}(p_n)_{\kappa_n} \Omega | \tilde{\Phi}(p_{n+1})_{\kappa_{n+1}} \dots \tilde{\Phi}(p_{n+m})_{\kappa_{n+m}} \Omega \rangle \\
&\qquad \times \overline{\tilde{\varphi}_n}(-p_1, \dots - p_n)_{\kappa_1 \dots \kappa_n} \, \tilde{\psi}_m(p_{n+1}, \dots p_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}} \\
&= \sum_{n,m} \sum_{(\kappa)_{n+m}} \int d(p)_{n+m} \, T_{n,m}((\mathbf{p})_{n+m})_{(\kappa)_{n+m}} \delta(p_1 + p_2 \dots + p_{n+m}) \\
&\qquad \times \prod_{j=1}^{n} \delta(p_j^2 - \lambda_{cj}^{-2}) \, (\omega_j - p_{j0}) \overline{\tilde{f}_n}(-p_1, \dots - p_n)_{\kappa_1 \dots \kappa_n} \\
&\qquad \times \prod_{j=n+1}^{n+m} \delta(p_j^2 - \lambda_{cj}^{-2}) \, (\omega_j + p_{j0}) \tilde{g}_m(p_{n+1}, \dots p_{n+m})_{\kappa_{n+1} \dots \kappa_{n+m}}
\end{aligned} \tag{114}$$

with $\underline{\varphi}$ and $\underline{\psi} \in \mathbf{H}_{\mathcal{P}}$ from (89),

$$\begin{aligned}
\tilde{\varphi}_n((p)_n) &= \prod_{j=1}^{n} (\omega_j + p_{j0}) \tilde{f}_n((p)_n)_{\kappa_1 \dots \kappa_n} \\
\tilde{\psi}_n((p)_n) &= \prod_{j=1}^{n} (\omega_j + p_{j0}) \tilde{g}_n((p)_n)_{\kappa_1 \dots \kappa_n}.
\end{aligned}$$

This scalar product is expressed for basis space $\mathcal{P}$ development in section 3.7.1 with the VEV from section 3.4. The φ_n, ψ_n are anti-local functions and $f_n, g_n \in \mathcal{S}(\mathbb{R}^{4n})$. $T_{n,m} \in \mathcal{S}'(\mathbb{R}^{3n+3m})$

and all energies p_{j0} are constrained to positive mass shells

$$p_{j0} = \omega_j$$

after reflection in four dimensions of the summation variables $(p)_n \mapsto (-p)_n$ in the $*$-dual function $\tilde{\varphi}_n$.

Isolating consideration to one function argument and one field component from a general state describing function $\tilde{\varphi}_n((p)_n)_{(\kappa)_n}$, let

$$\tilde{\varphi}(p_j) = (\omega_j + p_{j0})\tilde{f}(\mathbf{p}_j)$$

describe argument j for field component κ_j. Setting every $p_{j0} = \omega_j$ and noting the $(2\omega_j)^{-1}$ in the scalar product (114) from evaluation of $\delta(p_j^2 - \lambda_{cj}^{-2})$, the inverse Fourier transform is

$$\begin{aligned}
&\int \frac{dp}{(2\pi)^2}\, e^{ipx} e^{i\omega(\mathbf{p})\lambda_o} \tilde{f}(\omega(\mathbf{p}), \mathbf{p}) \\
&\qquad = \int \frac{dp_0}{(2\pi)^{\frac{1}{2}}}\, e^{ip_0(x_0 - \lambda)} \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega(\mathbf{p})(\lambda + \lambda_o)} \tilde{f}(\mathbf{p}) \qquad (115) \\
&\qquad = (2\pi)^{\frac{1}{2}} \delta(x_0 - \lambda) f(\lambda + \lambda_o, \mathbf{x})
\end{aligned}$$

with the designation

$$f(\lambda, \mathbf{x}) := \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega(\mathbf{p})\lambda} \tilde{f}(\mathbf{p}). \qquad (116)$$

Establishing the convention that every $p_{j0} = \omega_j$, the description of state at time $x_0 = 0$ is $f(\lambda_o, \mathbf{x})$, derived from

$$e^{i\omega(\mathbf{p})\lambda_o} \tilde{f}(\mathbf{p}),$$

the state description at time $x_0 = -\lambda_o$. $e^{i\omega(\mathbf{p})}$ is the time translation operator (93) that applies to argument j (92). The transform (116) is evaluated for convenient selections of packet functions $\tilde{f}(\mathbf{p})$ in appendix 6.10.

Finally, with the convention established, given a state describing function $\tilde{\varphi}_n((p)_n)$, the spatial support at time $\lambda = 0$ follows from the multivariable, momentum domain (three dimensional) inverse Fourier transform (16) of

$$\tilde{\varphi}_n((p)_n)_{(\kappa)_n} \in \mathcal{S}(\mathbb{R}^{4n}) \mapsto \prod_{j=1}^{n} \frac{\tilde{\varphi}_n((\omega(\mathbf{p}), \mathbf{p})_n)_{(\kappa)_n}}{2\omega_j} \in \mathcal{S}(\mathbb{R}^{3n}). \qquad (117)$$

For later times λ, the spatial support is the inverse Fourier transform (16) of

$$\prod_{j=1}^{n} e^{i\omega_j \lambda}\, \frac{\tilde{\varphi}_n((\omega(\mathbf{p}), \mathbf{p})_n)_{(\kappa)_n}}{2\omega_j} \in \mathcal{S}(\mathbb{R}^{3n}).$$

4.3 A nonrelativistic quantum-classical correspondence

From the nontrivial scattering amplitudes, including (100) and [31, 33, 35], it is evident that the VEV constructed in section 3.4 exhibit interaction. Interaction is also manifest in the finite interval, nonrelativistic transition amplitudes. In this section, a nonrelativistic correspondence of classical trajectories with the time evolution of state describing functions is developed.

Our perceptions of "macroscopic" and nonrelativistic quantum states satisfy classical dynamics. The quantum-classical correspondence introduced in sections 2 and 4.1 is that the isolated support of each argument of the state describing function $\varphi_n(\lambda)$ is well represented by a single location and momentum. These locations and momenta correspond with classical dynamical variables. Time evolution of the state describing function is represented by the temporal evolution of the classical dynamical variables. This correspondence of classical particle trajectories $\mathbf{u}_j(\lambda)$ with the evolution of a normalized state describing function is realized if

$$|U(\lambda)\hat{\varphi}_n(0)\rangle \approx e^{i\phi_I(\lambda)}|\hat{\varphi}_n(\lambda)\rangle \tag{118}$$

for times $\lambda > 0$. This quantum-classical correspondence applies for appropriate, normalized state descriptions such as (111). (118) adapts particle trajectories (106) to the evolution of state describing functions in quantum mechanics. Appropriate state describing functions have spatial and momentum support that is body-like: localized (102), well represented by a momentum (103), identifiable due to isolation (105), and nonrelativistic (109). Characterization of the evolution includes changes to the function $\varphi_n(\lambda)$ over the interval λ. Typically the breadth of the support of $\varphi_n((x)_n;\lambda)$ grows with λ from the unitary time evolution: classically, uncertainty in the initial momenta implies a spreading of the spatial support over time. Normalization is in the Hilbert space norm (20),

$$\hat{\varphi}_n(\lambda) := \frac{\varphi_n(\lambda)}{\|\varphi_n(\lambda)\|}. \tag{119}$$

Justified by Poincaré invariance of the scalar product (29), the reference time for the evolution in (118) is taken as zero without loss of generality. λ refers to time offset intervals. The quantum mechanical evolution of states follows a unitary (93) mapping $U(\lambda)$ (32) of the state describing functions.

$$U(\lambda)\varphi_n(0) = \varphi_n(-\lambda, x_1 - u_{1x}(0), y_1 - u_{1y}(0), z_1 - u_{1z}(0), \ldots; 0)$$

from (32) with functions $\varphi_n((0,\mathbf{x})_n;0)$ supported in a neighborhood of the spatial origin $(\mathbf{x})_n = 0$ with the designation for spatial vectors $\mathbf{x}_j = x_j, y_j, z_j$. (118) asserts that the nonrelativistic evolution of support follows trajectories $\mathbf{u}_j(\lambda) = u_{jx}(\lambda), u_{jy}(\lambda), u_{jz}(\lambda)$ that satisfy classical Newtonian dynamics,

$$\varphi_n(\lambda) = \varphi_n(0, x_1 - u_{1x}(\lambda), y_1 - u_{1y}(\lambda), z_1 - u_{1z}(\lambda), \ldots; \lambda).$$

In the description (111) of state describing functions with a classical correspondence, the displayed argument λ of $\varphi_n(\lambda)$ parameterizes the evolution of the trajectories $\mathbf{u}_k(\lambda)$ and a continuous deformation of $\varphi_n(0)$. The phase $\phi_I(\lambda)$ from (118) is determined by the state describing

functions $U(\lambda)\hat{\varphi}_n(0)$ and $\hat{\varphi}_n(\lambda)$, and is a purely quantum mechanical consideration following from the descriptions of states as rays in a complex Hilbert space. Including this phase and a deformation of the support of an appropriate state describing function, (118) represents the unitary evolution of the state describing function $\varphi_n(0)$ with the evolution of the classical dynamical variables $\mathbf{u}(\lambda)$.

A quantum-classical correspondence includes satisfaction of three considerations:

C.1) the likelihood that (118) is satisfied is near unity

C.2) the quantum-classical correspondence must persist: the supports of the functions (111) must remain localized (102), well represented by a momentum (103), isolated (105) and nonrelativistic (109)

C.3) identification of the corresponding trajectories $\mathbf{u}_j(\lambda)$.

If all three considerations are satisfied, then the trajectories $\mathbf{u}_k(\lambda)$ provide quantum-classical correspondences with the time evolution of the quantum state description at any time within the interval $(0, \lambda)$.

Position and momentum eigenfunctions are not suitable for quantum-classical correspondences. Position and momentum eigenfunctions are among the least classical-like by the criteria that location and momentum are both specified in a classical description. While $|\langle(\mathbf{y})_k|\varphi_n\rangle|^2$ provides the likelihood that the state described by φ_n will be jointly perceived near the $(\mathbf{y})_k$, $(\mathbf{y})_k$ does not describe a state with an evident classical interpretation: there is no indication of a momentum and k is not necessarily equal to n since $(\mathbf{y})_k$ is relativistic when sufficiently localized. Likelihoods $|\langle(\mathbf{y})_k|U(\lambda)\varphi_n\rangle|^2$ describe the temporal evolution of perceptions of location but not necessarily a correspondence with classical descriptions of particles. Similarly, there is no indication of location in the likelihoods $|\langle(\mathbf{q})_k|\tilde{\varphi}_n\rangle|^2$. Nevertheless, before the plane wave limit, likelihoods $|\langle(\mathbf{q})_k|\tilde{\varphi}_n\rangle|^2$ are useful for scattering since asymptotically the supports of the state describing functions are sufficiently spatially separated that cluster decomposition A.6 provides that localization (102) and isolation (105) apply to associate φ_n with corresponding classical particles. This correspondence from scattering does not apply for brief λ. For brief λ, (118) is the likelihood that a classical-like state is perceived with the evolution of nonrelativistic, classical-like initial state descriptions. In (118), the two states are described by the same particles with brief evolution of the momentum and location supports.

The approximate equality of state describing functions (118) is in the Hilbert space norm (20).

$$\left\| U(\lambda)\hat{\varphi}_n(0) - e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda) \right\|^2 \approx 0.$$

The norm is determined by the scalar products (21) of appropriate state describing functions (111) evaluated for VEV from section 3. The phase $\phi_I(\lambda)$ is an additive and homogeneous

function

$$\begin{aligned}\phi_I(a+b) &= \phi_I(a) + \phi_I(b)\\ \phi_I(0) &= 0.\end{aligned}$$

$\phi_I(\lambda)$ is determined to minimize the error in the approximation (118).

$$\begin{aligned}0 &\approx \left\| U(\lambda)\hat{\varphi}_n(0) - e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda)\right\|^2\\ &= 2 - 2\Re e\left(e^{i\phi_I(\lambda)}\langle U(\lambda)\hat{\varphi}_n(0)|\hat{\varphi}_n(\lambda)\rangle\right).\end{aligned}$$

$\|U(\lambda)\varphi_n(0)\| = \|\varphi_n(0)\|$ from Poincaré invariance of the scalar product. Error is minimized by maximization of the real component of the product of the phase factor and the scalar product. From

$$\Re e\left(e^{i\phi_I(\lambda)}\langle U(\lambda)\hat{\varphi}_n(0)|\hat{\varphi}_n(\lambda)\rangle\right) \leq |\langle U(\lambda)\hat{\varphi}_n(0)|\hat{\varphi}_n(\lambda)\rangle|,$$

polar decomposition of the scalar product determines the phase that achieves the maximum.

$$\langle U(\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle = e^{-i\phi_I(\lambda)}|\langle U(\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle|. \tag{120}$$

The Cauchy-Schwarz-Bunyakovsky inequality provides that

$$|\langle U(\lambda)\hat{\varphi}_n(0)|\hat{\varphi}_n(\lambda)\rangle| \leq 1. \tag{121}$$

It is convenient to designate the scalar product of normalized states

$$I(\lambda) := \frac{\langle U(\lambda)\varphi_n(0)|e^{i\phi_I(\lambda)}\varphi_n(\lambda)\rangle}{\langle\varphi_n(0)|\varphi_n(0)\rangle^{\frac{1}{2}}\,\langle\varphi_n(\lambda)|\varphi_n(\lambda)\rangle^{\frac{1}{2}}}. \tag{122}$$

$I(\lambda)$ is a composite function of differentiable trajectories $\mathbf{u}_k(\lambda)$ with $\mathbf{u}_k(0), \dot{\mathbf{u}}_k(0)$ considered as initial conditions for the corresponding classical trajectories. From Born's rule, $|I(\lambda)|^2$ is a likelihood, the likelihood that the state described by $U(\lambda)\varphi_n(0)$ is perceived as the state described by $\varphi_n(\lambda)$.

Continuous differentiability of $|I(\lambda)|^2$ suffices to satisfy C.1-3 for brief λ. A Taylor theorem polynomial in λ approximates the likelihood.

$$|I(\lambda)|^2 = |I(0)|^2 + \lambda\frac{d|I(0)|^2}{d\lambda} + \frac{1}{2}\lambda^2\frac{d^2|I(0)|^2}{d\lambda^2} + \ldots \tag{123}$$

and from the Cauchy-Schwarz-Bunyakovsky inequality (121), $\lambda = 0$ is recognized as a maxima with

$$\frac{d|I(0)|^2}{d\lambda} = 0.$$

From (122), $|I(0)|^2 = 1$ and for sufficiently brief λ,

$$\begin{aligned} |I(\lambda)|^2 &\approx 1 + \tfrac{1}{2}\lambda^2 \frac{d^2|I(0)|^2}{d\lambda^2} \\ &= 1 - \tfrac{1}{2}\lambda^2 \left| \frac{d^2|I(0)|^2}{d\lambda^2} \right| . \end{aligned} \tag{124}$$

For sufficiently brief intervals λ, the first derivative is proportional to λ.

$$\frac{d|I(\lambda)|^2}{d\lambda} \approx \lambda \frac{d^2|I(0)|^2}{d\lambda^2} .$$

To evaluate first derivatives of $|I(\lambda)|^2$ for $\lambda > 0$, the Taylor theorem polynomial expansion (123) must include order λ^2 terms. Extrapolation of the quantum-classical correspondence (118) from brief to more extended intervals is discussed in section 4.4.2.

Minimization of the first temporal derivative of $|I(\lambda)|^2$ provides that $|I(\lambda)| \approx 1$ for brief λ. The most likely corresponding classical trajectories $\mathbf{u}_j(\lambda)$ optimize

$$\frac{d|I(\lambda)|^2}{d\lambda} \approx 0 \tag{125}$$

for $\lambda > 0$. For the most likely trajectory, satisfaction of the quantum-classical correspondence (118) is not improved by any modification to the trajectory. For the optimal trajectory,

$$\begin{aligned} \frac{\partial|I(\lambda)|^2}{\partial \mathbf{u}_j(\lambda)} &= 0 \\ \frac{\partial|I(\lambda)|^2}{\partial \dot{\mathbf{u}}_j(\lambda)} &= 0 \end{aligned} \tag{126}$$

individually for each component of the $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$. Here, partial derivatives with respect to a spatial vector designates a gradient vector

$$\frac{\partial F(\mathbf{u})}{\partial \mathbf{u}} := \frac{\partial F(\mathbf{u})}{\partial u_x}, \frac{\partial F(\mathbf{u})}{\partial u_y}, \frac{\partial F(\mathbf{u})}{\partial u_z} .$$

The state describing functions $\tilde{\varphi}_n(\lambda)$ in (111) are composite functions including variations with $6n$ functions $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$. However, not all $6n$ functions are considered in (126). The center-of-momentum of the n interacting classical particles representative of $\varphi_n(\lambda)$ evolves as a single free body independently of the relative motion of the bodies. Independence applies in nonrelativistic approximation (109). Poincaré invariance (29) of the scalar product (21) and covariance (30) of state descriptions $\varphi_n(\lambda)$ provide that a Poincaré transformation equates any scalar product to the scalar product in a center-of-momentum reference frame for the n

trajectories $\mathbf{u}_j(\lambda)$ with momenta $\mathbf{w}_j(\lambda)$. In nonrelativistic approximation $\dot{\mathbf{u}}_j(\lambda) = \lambda_c \mathbf{w}_j(\lambda)$. In the center-of-momentum frame and for nonrelativistic momenta,

$$\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda) = 0, \quad \text{and} \qquad \sum_{j=1}^{n} m_{\kappa_j} \dot{\mathbf{u}}_j(\lambda) = 0.$$

Transformation to this frame is a boost described in appendix 6.11, and a translation to colocate the center-of-mass

$$\frac{\sum_{j=1}^{n} m_{\kappa_j} \mathbf{u}_j(\lambda)}{\sum_{j=1}^{n} m_{\kappa_j}}$$

with the origin of coordinates. As a consequence, 6 functions $\mathbf{u}_j(\lambda), \dot{\mathbf{u}}_j(\lambda)$ are linear combinations of the remaining $6(n-1)$ functions.

A chain rule expansion of $|I(\lambda)|^2$ includes partial derivatives with respect to the $n-1$ independent $\mathbf{u}_j(\lambda), \dot{\mathbf{u}}_j(\lambda)$.

$$\frac{d|I(\lambda)|^2}{d\lambda} = \sum_{j-1}^{n-1} \frac{\partial |I(\lambda)|^2}{d\mathbf{u}_j(\lambda)} \cdot \dot{\mathbf{u}}_j(\lambda) + \sum_{j-1}^{n-1} \frac{\partial |I(\lambda)|^2}{d\dot{\mathbf{u}}_j(\lambda)} \cdot \ddot{\mathbf{u}}_j(\lambda) + \frac{\partial |I(\lambda)|^2}{\partial \lambda} \tag{127}$$

with the $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$ held constant in evaluation of

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda}.$$

The $\mathbf{u}_j(\lambda)$ and $\dot{\mathbf{u}}_j(\lambda)$ are considered independent variables in $I(\lambda)$, $j \in \{1, n-1\}$. Then the trajectory selection criterion (125) is satisfaction of (126) with

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda} \approx 0. \tag{128}$$

In brief interval Taylor theorem polynomial expansions (123) for $|I(\lambda)|^2$ about $\lambda = 0$, the conditions (126) and (128) are satisfied to first order in brief λ. Then,

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = \mathcal{O}(\lambda^2)$$

satisfies (128) with β a component of $\mathbf{u}_j(\lambda)$, a component of $\dot{\mathbf{u}}_j(\lambda)$ or λ.

In section 4.4.5 below and within the fidelity of nonnrelativistic, brief interval, limited acceleration approximations,

$$\frac{d|I(\lambda)|^2}{d\beta} = \mathcal{O}(\lambda^2)$$

identically without constraint on $\mathbf{u}_j(\lambda)$ or $\dot{\mathbf{u}}_j(\lambda)$. Then, within the fidelity of the estimates selected to gain insight and minimize proliferation of terms, maximization of the likelihood does

not resolve a most likely trajectory. However, derivatives also relate the classical to quantum descriptions of the energy. This energy correspondence determines that the trajectories $\mathbf{u}_j(\lambda)$ derive from $-g/r$ pair potentials. Discussed in section 4.4.6 below and similarly to Schrödinger's development of the linear harmonic oscillator, [49] and section 4, the corresponding trajectories $\mathbf{u}_j(\lambda)$ are determined by observation.

The likelihood $|I(\lambda)|^2$ is the squared magnitude of the normalized scalar product (122). Derivatives of the likelihood follow from derivatives of the unnormalized scalar products. From (122) and the chain rule with β a component of $\mathbf{u}(\lambda)$, a component of $\dot{\mathbf{u}}(\lambda)$ or λ,

$$\frac{\partial I(\lambda)}{\partial \beta} = \frac{1}{\sqrt{S_0 S_\lambda}} \frac{\partial S_m}{\partial \beta} - \frac{S_m}{2 S_\lambda \sqrt{S_0 S_\lambda}} \frac{\partial S_\lambda}{\partial \beta}$$

with the designations

$$\begin{array}{rl} S_m & := \langle U(\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle \\ S_0 & := \langle \varphi_n(0)|\varphi_n(0)\rangle \\ S_\lambda & := \langle \varphi_n(\lambda)|\varphi_n(\lambda)\rangle. \end{array} \tag{129}$$

$I(\lambda) \neq 0$ and optimality (125) is satisfied if

$$\begin{aligned} \frac{\partial |I(\lambda)|^2}{\partial \beta} &= 2\mathbb{R}e\left(\overline{I(\lambda)}\,\frac{\partial I(\lambda)}{\partial \beta}\right) \\ &= 2\mathbb{R}e\left(\frac{\overline{S_m}}{S_0 S_\lambda}\,\frac{\partial S_m}{\partial \beta} - \frac{|S_m|^2}{2 S_0 S_\lambda^2}\,\frac{\partial S_\lambda}{\partial \beta}\right) \\ &= 0. \end{aligned}$$

$S_0, S_\lambda, S_m \neq 0$ and $S_0, S_\lambda, |S_m|^2 \in \mathbb{R}$. Then

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = 0$$

if and only if

$$\mathbb{R}e\left(\frac{2}{S_m}\,\frac{\partial S_m}{\partial \beta}\right) - \frac{1}{S_\lambda}\mathbb{R}e\left(\frac{\partial S_\lambda}{\partial \beta}\right) = 0. \tag{130}$$

Rather than determination of the quantum dynamics from canonical quantization of a classical dynamical model, the quantum dynamics is determined to satisfy general principles of quantum mechanics and relativity, and the classical dynamics corresponding to this quantum dynamics is derived. A classical correspondence is conditioned on characteristics of the quantum description of state. There is a quantum-classical correspondence if the quantum state description is "macroscopic," but if the quantum state description includes significant spatial overlaps or entanglement, then a quantum-classical correspondence does not apply.

Representations of the evolution of the state describing function by a single classical trajectory per body (118) deteriorate with increased propagation intervals. Considering the classical correspondence for each classical body, nearby initial conditions produce trajectories that diverge with time. With sufficient divergence of corresponding trajectories, association of the support of a state describing function with a classical body, satisfaction of C.2, is lost over time. Another perspective on the limited duration of the quantum-classical correspondence (118) is suggested by the Riemann-Lebesgue lemma [48]. The likelihood

$$|\langle U(\lambda)\varphi_n(0)|e^{i\phi_I(\lambda)}\varphi_n(\lambda)\rangle|^2$$

will asymptotically converge to zero for $\lambda \to \infty$ unless the phases $\exp(i\omega_k\lambda)$ from the Hamiltonian (92) are compensated by the state describing function $\varphi_n(\lambda)$. The phase $\phi_I(\lambda)$ does not suffice to compensate $\exp(i\omega_k\lambda)$ since $\phi_I(\lambda)$ does not vary with momenta $(\mathbf{p})_n$. An uncompensated phase proportional to λ leads to an asymptotically vanishing scalar product (122).

Persistent classical correspondences such as observations of planetary motions over great periods typically include recurring localizing observations. The relevant descriptions of state undergo recurring localization from interactions. The localizing observations are the result of scatter and emission of radiation, and perturbations of the motion of additional bodies. Inclusion of these effects on observed classical correspondences is an additional insight requiring further development.

4.4 Two body correspondence

This section is a substantial digression to evaluate the quantum-classical correspondence (118) for two-argument state describing functions. For two-argument state describing functions, the nonrelativistic correspondence is with two classical bodies. The two classical bodies are described by a freely evolving center-of-momentum and one independent trajectory $\mathbf{u}_1(\lambda)$. Nonrelativistic, brief interval, limited acceleration approximations for the scalar products (118) and first derivatives are evaluated. This example uses the VEV for a single neutral scalar field and appropriate state describing functions (111) from $\mathbf{H}_{\mathcal{P}}$.

A quantum-classical correspondences satisfies C.1-3 in section 4.3. Satisfaction of (118) maximizes the likelihood

$$|I(\lambda)|^2 = \frac{|\langle e^{-iH\lambda}\varphi_2(0)|\varphi_2(\lambda)\rangle|^2}{\langle\varphi_2(0)|\varphi_2(0)\rangle\langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle}. \tag{131}$$

Following (110) in section 4.1, appropriate two-argument state describing functions

$$\varphi_2((x)_2;\lambda)$$

have Fourier transforms

$$\tilde{\varphi}_2(p_1,p_2;\lambda) := \prod_{j=1}^{2} e^{-i\mathbf{p}_j\cdot\mathbf{u}_j(\lambda)}(p_{j0}+\omega_j)\ \tilde{f}_2(\mathbf{p}_1-\mathbf{w}_1(\lambda),\mathbf{p}_2-\mathbf{w}_2(\lambda);\lambda). \tag{132}$$

Poincaré invariance is exploited to express scalar products in the center-of-momentum coordinate frame determined by the corresponding classical locations $\mathbf{u}_j(\lambda)$ and momenta $\mathbf{w}_j(\lambda)$. In this center-of-momentum coordinate frame with the center-of-mass colocated with the origin,

$$\mathbf{u}_2(\lambda) = -\mathbf{u}_1(\lambda),$$

$\mathbf{w}_2(\lambda) = -\mathbf{w}_1(\lambda)$ and for nonrelativistic (109) momenta, $\lambda_c \mathbf{w}_2(\lambda) \approx \dot{\mathbf{u}}_j(\lambda)$.

4.4.1 Nonrelativistic approximation

While the quantum dynamics is relativistic, the quantum-classical correspondence discussed in section 4.3 applies for nonrelativistic, Newtonian classical dynamics. Scattering amplitudes, section 3.9, for example, the elastic scattering amplitude (100) for a single neutral scalar field, provide relativistic correspondences. In this section, nonrelativistic, reference frame-dependent approximations for the scalar products that determine the likelihoods $|I(\lambda)|^2$ from (131) are developed.

The form of $\tilde{f}_2$ in (132) is selected to remove consideration of the motion of the center-of-momentum from evaluation of the likelihoods $|I(\lambda)|^2$. In nonrelativistic approximation, length contraction and time dilation are negligible and the dynamics of the center-of-momentum decouples from internal motion. With the selected form, description of the motion of the center-of-momentum is factored from description of the relative motion of the two bodies. The selected $\tilde{f}_2$ separates in Jacobi coordinates in the center-of-momentum reference frame.

$$\tilde{f}_2(\mathbf{p}_1 - \mathbf{w}(\lambda), \mathbf{p}_2 + \mathbf{w}(\lambda); \lambda) := \tilde{f}_M(\mathbf{p}_1 + \mathbf{p}_2; \lambda) \tilde{f}_I(\mathbf{p}_1 - \mathbf{p}_2 - 2\mathbf{w}(\lambda); \lambda). \tag{133}$$

f_M describes the center-of-momentum, and f_I describes the relative motion of two corresponding classical bodies designated 1 and 2. The identification of classical bodies with arguments requires isolation (105). In the center-of-momentum reference frame, an abbreviated designation,

$$\mathbf{u}(\lambda) := \mathbf{u}_1(\lambda),$$

is substituted. Similarly, $\mathbf{w}(\lambda) := \mathbf{w}_1(\lambda)$. The supports of $\tilde{f}_M(\mathbf{p}; \lambda)$ and $\tilde{f}_I(\mathbf{p}; \lambda)$ are centered on the origins in both the space and momentum domains. Factors of $\omega_j + p_{j0}$ commute with the Hamiltonian (92) and then the time evolution of the state describing function is

$$e^{-iH\lambda} \tilde{f}_2((\mathbf{p} - \mathbf{w}(\lambda))_2; 0) = e^{-i(\omega_1 + \omega_2)\lambda} \, \tilde{f}_M(\mathbf{p}_1 + \mathbf{p}_2; 0) \tilde{f}_I(\mathbf{p}_1 - \mathbf{p}_2 - 2\mathbf{w}(0); 0).$$

Jacobi coordinates are

$$\mathbf{p}_1' := \mathbf{p}_1 + \mathbf{p}_2 \qquad \text{and} \qquad \mathbf{p}_2' := \mathbf{p}_1 - \mathbf{p}_2 \tag{134}$$

and then

$$\mathbf{p}_1 = \frac{\mathbf{p}_1' + \mathbf{p}_2'}{2} \quad \text{and} \qquad \mathbf{p}_2 = \frac{\mathbf{p}_1' - \mathbf{p}_2'}{2}.$$

With similar substitutions for $\mathbf{p}_3, \mathbf{p}_4$, the Jacobian for the coordinate transformation $(\mathbf{p})_4 \mapsto (\mathbf{p}')_4$ is $\left(\frac{1}{4}\right)^3$ for the four variables in three dimensional space.

Conservation of momentum separates in Jacobi coordinates. Factors of $\omega_j + p_{j0}$ after evaluation of mass shell delta functions in the state describing functions become approximately constant in nonrelativistic approximation, $\omega_j \approx \lambda_c^{-1}$ if $\lambda_c^2 \mathbf{p}_j^2 \ll 1$. The remaining separability considerations are the separability of the Hamiltonian and the conservation energy.

A separation of variables in the time translation (93) follows from nonrelativistic approximation (109) for the Hamiltonian (92). The Hamiltonian in the two-argument subspace is $\omega_1 + \omega_2$. Taylor theorem polynomial approximation results in

$$\begin{aligned} \omega_1 + \omega_2 &= \omega(\tfrac{\mathbf{p}_1' + \mathbf{p}_2'}{2}) + \omega(\tfrac{\mathbf{p}_1' - \mathbf{p}_2'}{2}) \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}_2') + \frac{\mathbf{p}_1'^2}{4\omega(\frac{1}{2}\mathbf{p}_2')} \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}_2') + \tfrac{1}{4}\lambda_c \mathbf{p}_1'^2 \end{aligned}$$

from (13) and with the nonrelativistic approximation (239) from appendix 6.13. $\lambda_c^2 {\mathbf{p}_1'}^2 \ll 1$. Then, for the state describing function (132) in nonrelativistic instances, time translation factors in Jacobi coordinates,

$$U(\lambda)\tilde{f}_2((\mathbf{p} - \mathbf{w}(\lambda))_2; 0) \approx \left(e^{-i\frac{1}{4}\lambda_c \mathbf{p}_1'^2 \lambda} \tilde{f}_M(\mathbf{p}_1'; 0)\right) \left(e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} \tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(0); 0)\right). \tag{135}$$

Nonrelativistic approximation of the Hamiltonian limits the duration of the interval λ in (93) that results in an accurate time translate. For negligible error,

$$\left(\omega_1 + \omega_2 - 2\omega(\frac{1}{2}\mathbf{p}_2') - \frac{1}{4}\lambda_c \mathbf{p}_1'^2\right) \lambda \ll \pi.$$

Indeed, the nonrelativistic approximations to ω_j are polynomials in $\mathbf{p}_j$ and are qualitatively different from Hamiltonians. $\sqrt{\lambda_c^{-2} - \Delta}$ is an anti-local operator [53] while powers of the Laplacian Δ are local.

Separation of variables in the energy conservation delta function also follows from nonrelativistic approximation, (240) in appendix 6.13.

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 \approx 2\omega(\frac{1}{2}\mathbf{p}_2') - 2\omega(\frac{1}{2}\mathbf{p}_4').$$

Conservation of momentum provides that $\mathbf{p}_1' = \mathbf{p}_3'$.

With time evolution used to define the deformation with time for the center-of-mass describing function,

$$\tilde{f}_M(\mathbf{p}; \lambda) = e^{-i\frac{1}{4}\lambda_c \mathbf{p}^2 \lambda} \tilde{f}_M(\mathbf{p}; 0), \tag{136}$$

nonrelativistic approximation of the Hamiltonians (135) and (136) provides that the state describing functions in the likelihood (131) include a common factor.

$$\begin{aligned} \tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;\lambda) &\approx \tilde{f}_M(\mathbf{p}_1';\lambda)\,\tilde{f}_I(\mathbf{p}_2'-2\mathbf{w}(\lambda);\lambda) \\ U(\lambda)\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_2;0) &\approx \tilde{f}_M(\mathbf{p}_1';\lambda)\,e^{-2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}\tilde{f}_I(\mathbf{p}_2'-2\mathbf{w}(0);0) \end{aligned} \tag{137}$$

are the two nonrelativistic descriptions of the evolution of the state describing functions in (131).

The cluster expansion (61) with (65) for the VEV of a single neutral scalar field provides that the VEV that define the scalar product for the two-argument function subspace are

$$\tilde{\mathcal{W}}_{2,2}((p)_4) = \tilde{\mathcal{U}}_{2,2}(((p)_4) + \tilde{W}_2(p_1,p_3)\tilde{W}_2(p_2,p_4) + \tilde{W}_2(p_1,p_4)\tilde{W}_2(p_2,p_3). \tag{138}$$

The connected functions are

$$\begin{aligned} \tilde{W}_2((p)_2) &= 2\sqrt{\omega_1\omega_2}\,\delta(\mathbf{p}_1+\mathbf{p}_2)\prod_{j=1}^{2}\delta(p_j^2-\lambda_c^{-2}) \\ \tilde{\mathcal{U}}_{2,2}((p)_4) &= c_4\,\delta(p_1+p_2+p_3+p_4)\prod_{j=1}^{4}\delta(p_j^2-\lambda_c^{-2}) \end{aligned}$$

from (37) and (54).

The frequency domain representation (114) of the scalar product with the single neutral scalar field VEV (138) and state describing functions (132) is

$$\begin{aligned} \langle\varphi_2(\lambda')|\varphi_2(\lambda)\rangle = \int d(\mathbf{p})_4\; & e^{i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{u}(\lambda')}\overline{\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda'))_2;\lambda')} \\ & \times e^{-i(\mathbf{p}_3-\mathbf{p}_4)\cdot\mathbf{u}(\lambda)}\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_{3,4};\lambda) \\ & \times\left\{\frac{4}{\lambda_c^2}\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4) + c_4\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\right\} \\ \approx \left(\tfrac{1}{4}\right)^3\int d(\mathbf{p}')_4\; & e^{i\mathbf{p}_2'\cdot\mathbf{u}(\lambda')}\overline{\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda'))_2;\lambda')}\;e^{-i\mathbf{p}_4'\cdot\mathbf{u}(\lambda)}\tilde{f}_2((\mathbf{p}-\mathbf{w}(\lambda))_{3,4};\lambda) \\ & \times\left\{\frac{4}{\lambda_c^2}\delta(\mathbf{p}_1'-\mathbf{p}_3')\delta(\mathbf{p}_2'-\mathbf{p}_4') + c_4\delta(2\omega(\tfrac{\mathbf{p}_2'}{2})-2\omega(\tfrac{\mathbf{p}_4'}{2}))\delta(\mathbf{p}_1'-\mathbf{p}_3')\right\} \end{aligned} \tag{139}$$

with the change to Jacobi coordinates (134) and in nonrelatistic approximation. The $\delta(\mathbf{p}_1'-\mathbf{p}_3')$ factor is conservation of momentum and is common among the remaining free field VEV and four-point connected VEV contributions. The

$$\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$$

"cross" term from the free field contribution to VEV is negligible for state describing functions that have a reliable quantum-classical correspondence. State describing functions that have a reliable quantum-classical correspondence satisfy isolation (105). The Pauli-Jordan two-point function has an exponential spatial decline in space-like directions characterized by $\|\mathbf{u}(\lambda)\|/\lambda_c$ and if separations $\|\mathbf{u}(\lambda)\|$ are large compared to λ_c, the contribution of the $\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)$ term is negligible.

Three scalar products contribute to $|I(\lambda)|^2$ in (131). Designate a generalized function in the scalar product (139) as

$$\begin{aligned} T_4(\mathbf{p}_2', \mathbf{p}_4') &:= \frac{1}{\lambda_c^2}\delta(\mathbf{p}_2'-\mathbf{p}_4') + \frac{c_4}{4}\delta(2\omega(\tfrac{\mathbf{p}_2'}{2})-2\omega(\tfrac{\mathbf{p}_4'}{2})) \\ &= \frac{1}{\lambda_c^2}\delta(\mathbf{p}_2'-\mathbf{p}_4') + \frac{c_4}{\lambda_c}\delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2) \end{aligned} \tag{140}$$

after the substitution

$$\delta(2\omega(\frac{\mathbf{p}_2'}{2}) - 2\omega(\frac{\mathbf{p}_4'}{2})) = 4\omega(\frac{\mathbf{p}_2'}{2})\delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2)$$

and the nonrelativistic approximation $\omega(\frac{\mathbf{p}_2'}{2}) \approx \lambda_c^{-1}$. Then, substitution of the state description (137) and VEV (140) into (139) results in

$$\begin{aligned} \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \approx \left(\tfrac{1}{4}\right)^2 \int d\mathbf{p}\, |\tilde{f}_M(\mathbf{p};\lambda)|^2 \int d\mathbf{p}_2' d\mathbf{p}_4'\, T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{i\mathbf{p}_2'\cdot\mathbf{u}(0)} e^{-i\mathbf{p}_4'\cdot\mathbf{u}(\lambda)}\, \overline{\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(0);0)}\, \tilde{f}_I(\mathbf{p}_4' - 2\mathbf{w}(\lambda);\lambda) \end{aligned}$$

denoted the mixed scalar product and

$$\begin{aligned} \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \approx \left(\tfrac{1}{4}\right)^2 \int d\mathbf{p}\, |\tilde{f}_M(\mathbf{p};\lambda)|^2 \int d\mathbf{p}_2' d\mathbf{p}_4'\, T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{i\mathbf{p}_2'\cdot\mathbf{u}(\lambda)} e^{-i\mathbf{p}_4'\cdot\mathbf{u}(\lambda)}\, \overline{\tilde{f}_I(\mathbf{p}_2' - 2\mathbf{w}(\lambda);\lambda)}\, \tilde{f}_I(\mathbf{p}_4' - 2\mathbf{w}(\lambda);\lambda) \end{aligned}$$

provides the square of both norms, $\|\varphi_2(0)\|$ and $\|\varphi_2(\lambda)\|$. From unitary realization of time translation $U(\lambda)$, it follows that $\|U(\lambda)\varphi_2(0)\| = \|\varphi_2(0)\|$. Due to the unimodular phase in (136), the summation

$$\int d\mathbf{p}\, |\tilde{f}_M(\mathbf{p};\lambda)|^2 = \int d\mathbf{p}\, |\tilde{f}_M(\mathbf{p};0)|^2$$

does not vary with time λ and becomes a common factor of the three scalar products of interest. Designate the common factor in the three scalar products composing the likelihood (131)

$$a_g := \left(\frac{1}{4}\right)^2 \int d\mathbf{p}\, |\tilde{f}_M(\mathbf{p};\lambda)|^2. \tag{141}$$

Common factors do not contribute to the likelihood (131). Also, since likelihood depends only on the magnitude of the mixed scalar product, unimodular factors $e^{i\phi_I(\lambda)}$ do not contribute.

A Taylor theorem polynomial expansion for $2\omega(\frac{\mathbf{p}_2'}{2})$ is

$$2\omega(\frac{\mathbf{p}_2'}{2}) \approx 2\omega(\mathbf{w}(\lambda)) + \frac{\mathbf{w}(\lambda)\cdot(\mathbf{p}_2' - 2\mathbf{w}(\lambda))}{\omega(\mathbf{w}(\lambda))} + \frac{(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}{4\omega(\mathbf{w}(\lambda))}$$

from (239) in appendix 6.13. Expansion is about the dominant momentum $2\mathbf{w}(\lambda)$ in the support over $\mathbf{p}_2'$. The support of the description of the internals is nonrelativistic if

$$\|\mathbf{p}_2' - 2\mathbf{w}(\lambda)\| \ll \lambda_c^{-1} \leq \omega(\mathbf{w}(\lambda))$$

within the dominant support of $\tilde{f}_I$ and the corresponding classical trajectory is nonrelativistic if $\|\mathbf{w}(\lambda)\| \ll \lambda_c^{-1}$. For nonrelativistic momenta, the Hamiltonian that describes the evolution of the internals is

$$\begin{aligned} 2\omega(\tfrac{\mathbf{p}_2'}{2}) &\approx 2\lambda_c^{-1} + \lambda_c\mathbf{w}(\lambda)^2 + \lambda_c\mathbf{w}(\lambda)\cdot(\mathbf{p}_2' - 2\mathbf{w}(\lambda)) + \frac{\lambda_c}{4}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2 \\ &= 2\lambda_c^{-1} + \frac{\lambda_c}{4}\mathbf{p}_2'^2 \end{aligned} \tag{142}$$

to second order in small quantities $\lambda_c\|\mathbf{p}_2' - 2\mathbf{w}(\lambda)\| \ll 1$ and $\lambda_c\|\mathbf{w}(\lambda)\| \ll 1$.

With nonrelativistic approximation for the Hamiltonian, time translations of a Gaussian internals describing function remains in family. Selection of a Gaussian to describe the internals,

$$\tilde{f}_I(\mathbf{p};\lambda) := e^{2i\mathbf{u}(\lambda)\cdot\mathbf{w}(\lambda)}\exp(-L(\lambda)^2\mathbf{p}^2), \tag{143}$$

satisfies localization (102) with a dominant momentum (103), is spherically symmetric, and with the nonrelativistic approximation, $U(\lambda)\tilde{f}_I(\mathbf{p};0)$ is also Gaussian. $L(\lambda)$ is a complex length characterizing the breadth of the spatial support. In (143)

$$\mathbb{R}e(L(\lambda)^2) > 0.$$

Real $L(0)^2$ are the most classical-like state describing functions, meeting the Heisenberg uncertainty lower bound for simultaneous knowledge of location and momentum: complex $L(0)^2$ exceed the lower uncertainty bound with additional spreading in location for a constant spread in momentum. The translated functions

$$e^{-i\mathbf{p}\cdot\mathbf{u}(\lambda)}\,\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(\lambda);\lambda) = \exp\left(-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 - i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)\right)$$

have inverse Fourier transforms

$$\left(\frac{1}{2L(\lambda)^2}\right)^{\frac{3}{2}}\exp\left(-\frac{(\mathbf{x}-\mathbf{u}(\lambda))^2}{4L(\lambda)^2} + 2i\mathbf{w}(\lambda)\cdot\mathbf{x}\right).$$

To satisfy a nonrelativistic quantum-classical correspondence, the support spread parameter $L(\lambda)^2$ is constrained by both upper and lower bounds. An upper bound

$$\frac{|L(\lambda)|^4}{\mathbb{R}e(L(\lambda)^2)} \ll \mathbf{u}^2$$

limits spatial extent to isolate support from the support of the other spatial argument (105). A lower bound suppresses support on relativistic momenta (109). The dominant support of the internals describing function (143) satisfies

$$\|\mathbf{p} - 2\mathbf{w}(\lambda)\| \leq \frac{K}{(\mathbb{R}e(L(\lambda)^2))^{\frac{1}{2}}}$$

for a real $K \approx 10$. If

$$\|\mathbf{p} - 2\mathbf{w}(\lambda)\| \leq \frac{K}{(\mathbb{R}e(L(\lambda)^2))^{\frac{1}{2}}} \ll \frac{1}{\lambda_c}$$

and if the classical trajectory is nonrelativistic, $\lambda_c\|\mathbf{w}\lambda\| \ll 1$, then the dominant support is nonrelativistic. For a nonrelativistically supported function (143),

$$\mathbb{R}e(L_0(\lambda)^2) \gg \lambda_c^2. \tag{144}$$

Substitution of the Gaussian internals describing function (143) into the scalar product results in

$$\begin{aligned}\langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2}\end{aligned} \tag{145}$$

for the mixed scalar product and

$$\begin{aligned}\langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \;\; \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ \times e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2}\end{aligned} \tag{146}$$

for the squared norms. Notation includes the single scalar field VEV (140) and the common factor (141) that includes the description of the center-of-momentum.

With $\mathbb{R}e(L(\lambda)^2) > 0$ and $\mathbf{u}(\lambda), \mathbf{w}(\lambda) \in \mathbb{R}$, after evaluation of the delta functions, the rapid decline of Gaussian functions ensures convergence of the scalar products (145) and (146). From the dominated convergence theorem, derivatives of the scalar products are summations of the derivatives of state describing functions. With the designations (129), $S_m = \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle$

from (145) and $S_\lambda = \langle\varphi_2(\lambda)|\varphi_2(\lambda)\rangle$ from (146), the derivatives of the scalar products with respect to the components of $\mathbf{u}(\lambda)$ are

$$\frac{\partial S_m}{\partial \mathbf{u}(\lambda)} \approx -ia_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; (\mathbf{p}_4' - 2\mathbf{w}(\lambda)) \\ \times e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)}\; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{147}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \mathbf{u}(\lambda)} \approx ia_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; (\mathbf{p}_2' - \mathbf{p}_4') \\ \times e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{148}$$

for the $\lambda > 0$ squared norms. The $\lambda = 0$ squared norms are independent of $\mathbf{u}(\lambda)$, $\dot{\mathbf{u}}(\lambda)$ and λ.

Similarly, the partial derivatives of the scalar products with respect to the components of $\dot{\mathbf{u}}(\lambda) = \lambda_c \mathbf{w}(\lambda)$ in the nonrelativistic approximation (109) are

$$\frac{\partial S_m}{\partial \dot{\mathbf{u}}(\lambda)} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; \left(2i\lambda_c^{-1}\mathbf{u}(\lambda) + 4\lambda_c^{-1} L(\lambda)^2 (\mathbf{p}_4' - 2\mathbf{w}(\lambda))\right) \\ \times e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)}\; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2}\; e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \tag{149}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \dot{\mathbf{u}}(\lambda)} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; 4\lambda_c^{-1} \left(\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda)) + L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\right) \\ \times e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{150}$$

for the $\lambda > 0$ squared norms.

Finally, the partial derivatives of the scalar products with respect to λ with $\mathbf{u}(\lambda)$ and $\dot{\mathbf{u}}(\lambda)$ held constant are

$$\frac{\partial S_m}{\partial \lambda} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; \left(2i\omega(\tfrac{\mathbf{p}_2'}{2}) - \frac{dL(\lambda)^2}{d\lambda}(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2\right) \\ \times e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}\; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{151}$$

for the mixed scalar product and

$$\frac{\partial S_\lambda}{\partial \lambda} \approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4'\; T_4(\mathbf{p}_2', \mathbf{p}_4')\; \left(-\frac{d\overline{L(\lambda)^2}}{d\lambda}((\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2 - \frac{dL(\lambda)^2}{d\lambda}(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2)\right) \\ \times e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(\lambda)} e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2}\; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \tag{152}$$

for the $\lambda > 0$ squared norms. In nonrelativistic approximation, $\omega(\frac{\mathbf{p}_2'}{2}) \approx \lambda_c^{-1}$.

4.4.2 Brief interval and limited acceleration approximations

Approximations that apply for sufficiently brief intervals λ enable evaluation of the expressions for the scalar products and their first derivatives (145)-(152) as elementary functions. Except for the nonrelativistic approximation of the Hamiltonian that limits the interval λ, only nonrelativistic approximations have been applied to this point. In this section, brief interval and iimited acceleration approximations are developed.

If the quantum-classical correspondence is examined for sufficiently brief intervals $\lambda > 0$, then Taylor theorem polynomial approximation to first order in λ suffices.

$$\begin{aligned} \mathbf{u}(\lambda) &\approx \mathbf{u}(0) + \lambda\dot{\mathbf{u}}(0) \\ \mathbf{w}(\lambda) &\approx \mathbf{w}(0) + \lambda\dot{\mathbf{w}}(0) \end{aligned} \tag{153}$$

and for nonrelativistic momenta, $\lambda_c \mathbf{w}(\lambda) \approx \dot{\mathbf{u}}(\lambda)$. The Gaussian state describing functions (143),

$$e^{-i\mathbf{p}\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(\lambda); \lambda)$$

and

$$e^{2i\omega(\frac{\mathbf{p}}{2})}e^{-i\mathbf{p}\cdot\mathbf{u}(0)}\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(0); 0)$$

are simply related to $\mathcal{O}(\lambda^2)$ for nonrelativistic momenta (109).

$$\begin{aligned} e^{-i\mathbf{p}\cdot\mathbf{u}(\lambda)}\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(\lambda); \lambda) &= e^{-i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}e^{-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2} \\ &\approx e^{-2i\omega(\frac{\mathbf{p}}{2})\lambda}e^{-i\phi_I(\lambda)}\ e^{-i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2}\ e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} \\ &= e^{2i\omega(\frac{\mathbf{p}}{2})}e^{-i\phi_I(\lambda)}\ e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}\ e^{-i\mathbf{p}\cdot\mathbf{u}(0)}\tilde{f}_I(\mathbf{p} - 2\mathbf{w}(0); 0). \end{aligned} \tag{154}$$

The relationship (154) includes an envelope evolution correction factor $e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}$, the unimodular factor $e^{-i\phi_I(\lambda)}$ and the temporal translation $e^{2i\omega(\frac{\mathbf{p}}{2})}$. The brief interval approximation (154) follows from Taylor theorem expansion if the $\phi_I(\lambda)$ from (118) is

$$\phi_I(\lambda) = -(2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0))\frac{\lambda}{\lambda_c}, \tag{155}$$

and if

$$L(\lambda)^2 = L(0)^2 - i\frac{\lambda_c}{4}\lambda. \tag{156}$$

This $L(\lambda)^2$ from (143) describes the support spread from nonrelativistic free propagation of a Gaussian function [24] in Jacobi coordinates (134). The phase (155) corresponds to the classical energy of two nonrelativistic particles.

$$2mc^2 + \text{K.E.} + \text{V} = mc^2\ (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0)).$$

$\lambda_c mc^2 = \hbar c$. The rest mass energy is 2, the kinetic energy of the two particles is $\dot{\mathbf{u}}(0)^2$, and the pair potential V is identified in more familar form in section 4.4.6 below. If the trajectories satisfy Newton's equation of motion, this energy is independent of λ.

Demonstration of (154) begins with Taylor polynomial approximation (142) of the Hamiltonian.

$$\begin{aligned}
&2i\omega(\tfrac{\mathbf{p}}{2})\lambda + i(\mathbf{p}-2\mathbf{w}(0)) \cdot \mathbf{u}(0) + L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 \\
&\quad \approx i\lambda(2\lambda_c^{-1} + \lambda_c \mathbf{w}(0)^2 + \frac{\lambda_c}{4}\mathbf{p}^2) + i(\mathbf{p}-2\mathbf{w}(0)) \cdot \mathbf{u}(0) + L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 \\
&\quad \approx i\lambda(2\lambda_c^{-1} + \lambda_c \mathbf{w}(0)^2 + \frac{\lambda_c}{4}\mathbf{p}^2) + i(\mathbf{p}-2\mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) + L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 \\
&\qquad -i\lambda(\mathbf{p}-2\mathbf{w}(0))\dot{\mathbf{u}}(0) + 2i\lambda\dot{\mathbf{w}}(0) \cdot \mathbf{u}(0)
\end{aligned}$$

from substitutions of the linear approximations (153) and neglect of $\mathcal{O}(\lambda^2)$ terms. From the identity in the nonrelativistic approximation for energy (142), the nonrelativistic relation $\lambda_c \mathbf{w} \approx \dot{\mathbf{u}}$, and substitution of the linear expansions (153),

$$\begin{aligned}
&2i\omega(\tfrac{\mathbf{p}}{2})\lambda + i(\mathbf{p}-2\mathbf{w}(0)) \cdot \mathbf{u}(0) + L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 \\
&\quad \approx i\frac{\lambda}{\lambda_c}(2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0)) + i\lambda((\mathbf{p} - 2\mathbf{w}(\lambda)) \cdot \dot{\mathbf{u}}(\lambda) + \frac{\lambda_c}{4}(\mathbf{p} - 2\mathbf{w}(\lambda))^2) \\
&\qquad +i(\mathbf{p}-2\mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) + L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 - i\lambda(\mathbf{p}-2\mathbf{w}(0))\dot{\mathbf{u}}(\lambda) \\
&\quad \approx i\frac{\lambda}{\lambda_c}(2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0) \cdot \mathbf{u}(0)) + i(\mathbf{p}-2\mathbf{w}(\lambda)) \cdot \mathbf{u}(\lambda) + i\lambda\frac{\lambda_c}{4}(\mathbf{p} - 2\mathbf{w}(\lambda))^2) \\
&\qquad +L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2
\end{aligned}$$

neglecting $\mathcal{O}(\lambda^2)$ terms. The linear expansion (153) and neglecting $\mathcal{O}(\lambda^2)$ terms provides that

$$L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2 = L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2 - 4L(0)^2(\mathbf{p}-2\mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)\lambda + \mathcal{O}(\lambda^2).$$

Finally, the identifications (155) and (156) demonstrate (154).

Neglecting terms of $\mathcal{O}(\lambda^2)$ in the brief interval expansion, the resulting envelope evolution correction factor does not vary with time for any of the parameters $L(\lambda)^2$, $\mathbf{u}(\lambda)$, nor $\dot{\mathbf{u}}(\lambda)$.

$$\begin{aligned}
e^{-4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda} &\approx e^{-4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot(\dot{\mathbf{w}}(0)+\lambda\ddot{\mathbf{w}}(0))\lambda} \\
&\approx e^{-4L(0)^2(\mathbf{p}-2\mathbf{w}(0)-2\lambda\dot{\mathbf{w}}(0))\cdot\dot{\mathbf{w}}(\lambda)\lambda} \\
&\approx e^{-4(L(0)^2-i\frac{\lambda_c}{4}\lambda)(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\dot{\mathbf{w}}(\lambda)\lambda} \\
&= e^{-4L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\dot{\mathbf{w}}(\lambda)\lambda}.
\end{aligned} \tag{157}$$

An acceleration limitation is convenient to analyze the quantum-classical correspondence (118) by justifying neglect of the envelope evolution correction factor. The envelope evolution

correction in the approximation (154) is negligible either with an acceleration limit, or after application of dominant momentum (103) in (167) below. From the Cauchy-Schwarz-Bunyakovsky inequality,

$$|(\mathbf{p}-2\mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)| \leq \|\mathbf{p}-2\mathbf{w}(0)\| \, \|\dot{\mathbf{w}}(0)\|.$$

If the initial acceleration in the brief interval approximation is limited,

$$\|\dot{\mathbf{w}}(0)\| \leq \frac{\epsilon}{\mathbf{u}^2}, \tag{158}$$

then

$$|(\mathbf{p}-2\mathbf{w}(0)) \cdot \dot{\mathbf{w}}(0)| \leq \|\mathbf{p}-2\mathbf{w}(0)\| \frac{\epsilon}{\mathbf{u}^2}$$

and the envelope evolution correction factor $e^{4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}$ is negligible compared to the envelope,

$$4\mathbb{R}e(L(0)^2)\|\mathbf{p}-2\mathbf{w}(0)\|\frac{\epsilon}{\mathbf{u}^2}\lambda \ll \mathbb{R}e(L(0)^2)(\mathbf{p}-2\mathbf{w}(0))^2,$$

as long as

$$\frac{4\epsilon\lambda}{\mathbf{u}^2} \ll \|\mathbf{p}-2\mathbf{w}(0)\|.$$

The support of the summation evaluating a scalar product includes $\mathbf{p} \approx 2\mathbf{w}(0)$ but the envelope evolution correction is small in the neighborhood of these points. To approximate scalar products, the bound need only apply for $\mathbf{p}$ where the envelope deviates significantly from unity, $\mathbf{p}$ that satisfy

$$\sqrt{\mathbb{R}e(L(0)^2)} \, \|\mathbf{p}-2\mathbf{w}(0)\| \geq \epsilon.$$

An accurate approximation for the scalar product results if $\epsilon \ll 1$. Then, the envelope evolution factor is well approximated by unity if the interval λ is bounded,

$$\lambda \ll \frac{\mathbf{u}^2}{4\sqrt{\mathbb{R}e(L(0)^2)}}.$$

This upper bound on λ increases with greater initial separations $\|\mathbf{u}\|$. With the acceleration limit (158) and for the sufficiently brief intervals λ, the envelope evolution correction factor (157) contributed by substitutions of (159) are negligible,

$$e^{4L(0)^2\mathbf{q}_j\cdot\dot{\mathbf{w}}(0)\lambda} \approx 1.$$

Finally, the nonrelativistic, brief interval, limited acceleration approximation is

$$e^{-i(\mathbf{p}-2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)}e^{-L(\lambda)^2(\mathbf{p}-2\mathbf{w}(\lambda))^2} \approx e^{-i\phi_I(\lambda)}e^{-2i\omega(\frac{\mathbf{p}}{2})\lambda} \, e^{-i(\mathbf{p}-2\mathbf{w}(0))\cdot\mathbf{u}(0)}e^{-L(0)^2(\mathbf{p}-2\mathbf{w}(0))^2}. \tag{159}$$

Substitution of the brief interval, limited acceleration approximation (159) into the mixed scalar product (145) and squared norms (146) result in

$$\begin{aligned}
S_m &= \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} \; e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \\
&\quad \times e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \\
&\quad \times e^{-i\phi_I(\lambda)} e^{-2i\omega(\frac{\mathbf{p}_4'}{2})\lambda} \; e^{-i(\mathbf{p}_4' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2}
\end{aligned}$$

and

$$\begin{aligned}
S_\lambda &= \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{i(\mathbf{p}_2' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-\overline{L(\lambda)^2}(\mathbf{p}_2' - 2\mathbf{w}(\lambda))^2} \\
&\quad \times e^{-i(\mathbf{p}_4' - 2\mathbf{w}(\lambda))\cdot\mathbf{u}(\lambda)} \; e^{-L(\lambda)^2(\mathbf{p}_4' - 2\mathbf{w}(\lambda))^2} \\
&\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{i\phi_I(\lambda)} e^{2i\omega(\frac{\mathbf{p}_2'}{2})\lambda} \; e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} \\
&\quad \times e^{-i\phi_I(\lambda)} e^{-2i\omega(\frac{\mathbf{p}_4'}{2})\lambda} \; e^{-i(\mathbf{p}_4' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2}.
\end{aligned}$$

The Hamiltonian derived factors $e^{2i(\omega(\frac{\mathbf{p}_2'}{2}) - \omega(\frac{\mathbf{p}_4'}{2}))\lambda} = 1$ due either to conservation of momentum, $\mathbf{p}_2' = \mathbf{p}_4'$, from the free field VEV or conservation of energy,

$$2\omega(\frac{\mathbf{p}_2'}{2}) = 2\omega(\frac{\mathbf{p}_4'}{2})$$

from the connected VEV. The unimodular phase factor $e^{-i\phi_i(\lambda)}$ contributed by substitution of (159) distributes out of the summation and therefore does not contribute to likelihood (131). The approximations for the scalar products neglect terms of $\mathcal{O}(\lambda^2)$, are to second order in the nonrelativistic approximation (109), and apply for limited accelerations (158). The approximations are to second order in the small quantities

$$\lambda_c \|\mathbf{p}_2' - 2\mathbf{w}(0)\|, \lambda_c \|\mathbf{p}_4' - 2\mathbf{w}(0)\| \text{ and } \lambda_c \|\mathbf{w}(0)\| \ll 1.$$

Factors

$$e^{i(\mathbf{p}_2' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} e^{-i(\mathbf{p}_4' - 2\mathbf{w}(0))\cdot\mathbf{u}(0)} = e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)}$$

from the common time of the brief interval approximation in (159). Simplification results in

$$\begin{aligned}
S_m &\approx a_g e^{-i\phi_I(\lambda)} \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2} \\
S_\lambda &\approx a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; e^{i(\mathbf{p}_2' - \mathbf{p}_4')\cdot\mathbf{u}(0)} e^{-\overline{L(0)^2}(\mathbf{p}_2' - 2\mathbf{w}(0))^2} e^{-L(0)^2(\mathbf{p}_4' - 2\mathbf{w}(0))^2}
\end{aligned} \tag{160}$$

That the scalar products

$$S_m = S_\lambda = S_0$$

at $\lambda = 0$ follows from (118).

4.4.3 A functional $Q(F(\mathbf{q}_2, \mathbf{q}_4))$

In this section, the nonrelativistic, brief interval approximations for the scalar products and their derivatives (145)-(152) are expressed using a convenient functional.

To abbreviate notation here and in sections 4.4.4-4.4.6, designations for the initial conditions of the classical trajectories are abbreviated

$$\mathbf{u} := \mathbf{u}(0), \quad \dot{\mathbf{u}} := \dot{\mathbf{u}}(0), \quad \mathbf{w} := \mathbf{w}(0), \quad \text{and} \quad \dot{\mathbf{w}} := \dot{\mathbf{w}}(0). \tag{161}$$

For brief $\lambda > 0$, time evolution of the classical dynamical variables is linearly approximated (153).

The functional Q maps multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ of two spatial vectors $\mathbf{q}_2, \mathbf{q}_4 \in \mathbb{R}^3$ to $\mathbb{C}$. the function is

$$\begin{aligned} Q(F(\mathbf{q}_2, \mathbf{q}_4)) := a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \; F(\mathbf{p}_2' - 2\mathbf{w}, \mathbf{p}_4' - 2\mathbf{w}) \\ \times e^{\mathbf{b}_2 \cdot \mathbf{p}_2' + \mathbf{b}_4 \cdot \mathbf{p}_4'} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}. \end{aligned} \tag{162}$$

$Q(F(\mathbf{q}_2, \mathbf{q}_4))$ results in the brief interval, nonrelativistic approximations to the scalar products (160) and their first derivatives (145)-(152) using the approximation (159) from section 4.4.2. The generalized function $T_4(\mathbf{p}_2', \mathbf{p}_4')$ is from the VEV (140) and a_g is from (141). σ^2, $\mathbf{b}_2$ and $\mathbf{b}_4$ are independent complex parameters. $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is linear.

$$Q(\alpha F_1 + \beta F_2) = \alpha Q(F_1) + \beta Q(F_2)$$

for $\alpha, \beta \in \mathbb{C}$ and multinomials $F_!(\mathbf{q}_2, \mathbf{q}_4), F_2(\mathbf{q}_2, \mathbf{q}_4)$. $\mathbf{q}_j$ is an abbbreviated notation indicating a factor of $\mathbf{p}_j' - 2\mathbf{w}$ in the summation. After evaluation of the momentum and energy conservation generalized functions from T_4 (140), the summation (162) is absolutely convergent. The dominated convergence theorem justifies interchange of summation and differentiation. Then

$$\begin{aligned} Q(F(\mathbf{q}_2, \mathbf{q}_4)) &= a_g \int d\mathbf{p}_2' d\mathbf{p}_4' \; T_4(\mathbf{p}_2', \mathbf{p}_4') \\ &\quad \times F(\frac{\partial}{\partial \mathbf{b}_2} - 2\mathbf{w}, \frac{\partial}{\partial \mathbf{b}_4} - 2\mathbf{w}) \; e^{\mathbf{b}_2 \cdot \mathbf{p}_2' + \mathbf{b}_4 \cdot \mathbf{p}_4'} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2} \\ &= F(\frac{\partial}{\partial \mathbf{b}_2} - 2\mathbf{w}, \frac{\partial}{\partial \mathbf{b}_4} - 2\mathbf{w}) \, Q(1). \end{aligned} \tag{163}$$

The values of physical interest result from substitution of the initial location $\mathbf{u}$ and complex squared length $L(0)^2$,

$$\begin{aligned} \sigma^2 &:= L(0)^2 \\ \mathbf{b}_2 &:= i\mathbf{u} \\ \mathbf{b}_4 &:= -i\mathbf{u}. \end{aligned} \tag{164}$$

$Q(1)$ is evaluated in elementary functions in section 4.4.4.

If $\mathbf{b}_2 = \overline{\mathbf{b}_4}$, then $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is transpose conjugate symmetric. Transpose symmetry of the real $T_4(\mathbf{p}'_2, \mathbf{p}'_4)$ (140) and relabeling of the summation variables $\mathbf{p}'_2 \leftrightarrow \mathbf{p}'_4$ demonstrates that

$$Q(F(\mathbf{q}_2, \mathbf{q}_4)) = \overline{Q(\,\overline{F(\mathbf{q}_4, \mathbf{q}_2)}\,)}. \tag{165}$$

From (165), $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is real for real, transpose symmetric multinomials,

$$F(\mathbf{q}_2, \mathbf{q}_4) = F(\mathbf{q}_4, \mathbf{q}_2) = \overline{F(\mathbf{q}_2, \mathbf{q}_4)}, \tag{166}$$

and $Q(F(\mathbf{q}_2, \mathbf{q}_4))$ is imaginary for real, transpose antisymmetric multinomials,

$$F(\mathbf{q}_2, \mathbf{q}_4) = -F(\mathbf{q}_4, \mathbf{q}_2) = \overline{F(\mathbf{q}_2, \mathbf{q}_4)}.$$

Without the acceleration limit (158), the brief interval approximation (159) retains envelope evolution factors (157), $e^{-4L(0)^2(\mathbf{p}-2\mathbf{w}(0))\cdot\dot{\mathbf{w}}(0)\lambda}$. Without an acceleration limit and neglecting the overall phase $e^{-i\phi_I(\lambda)}$ that does not contribute to likelihood, substitution of (162) results in

$$\begin{aligned} \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle &\approx Q(e^{4\lambda L(0)^2 \mathbf{q}_4\cdot\dot{\mathbf{w}}}) \\ &\approx Q(1) + 4\lambda L(0)^2\, Q(\mathbf{q}_4\cdot\dot{\mathbf{w}}) \end{aligned}$$

for the mixed scalar product and

$$\begin{aligned} \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle &\approx Q(e^{4\lambda\overline{L(0)^2}\mathbf{q}_2\cdot\dot{\mathbf{w}}} e^{4\lambda(L(0)^2\mathbf{q}_4\cdot\dot{\mathbf{w}}}) \\ &\approx Q(1) + 4\lambda\overline{L(0)^2}Q(\mathbf{q}_2\cdot\dot{\mathbf{w}}) + 4\lambda L(0)^2 Q(\mathbf{q}_4\cdot\dot{\mathbf{w}}) \end{aligned}$$

for the squared norms. These expression include the abbreviated notation (161) for the initial trajectory. From conjugate symmetry (165),

$$Q(\mathbf{q}_2\cdot\dot{\mathbf{w}}) = \overline{Q(\mathbf{q}_4\cdot\dot{\mathbf{w}})}.$$

First order in λ Taylor theorem polynomial approximation of the envelope evolution factors (157), linearity and the conjugate symmetry (165) of $Q(F)$ provide that the likelihood (131) is

$$\begin{aligned} |I(\lambda)|^2 &\approx \frac{|Q(1) + 4\lambda L(0)^2\, Q(\mathbf{q}_4\cdot\dot{\mathbf{w}})|^2}{Q(1)(Q(1) + 4\lambda L(0)^2 Q(\mathbf{q}_4\cdot\dot{\mathbf{w}}) + 4\lambda\,\overline{L(0)^2 Q(\mathbf{q}_4\cdot\dot{\mathbf{w}})})} \\ &= 1 + \mathcal{O}(\lambda^2). \end{aligned} \tag{167}$$

This brief interval, nonrelativistic approximation follows from the more general result (124). $Q(1)$ is real from (166). This result also provides that to $\mathcal{O}(\lambda^2)$, the derivatives (145)-(152) of the likelihood $|I(\lambda)|^2$ are invariant to whether the envelope evolution correction factor is included if a dominant momentum approximation (103) suffices.

Brief interval, nonrelativistic and limited acceleration approximations for the scalar products and their derivatives (145)-(152) are compactly expressed using the functional $Q(F(\mathbf{q}_2, \mathbf{q}_4))$. From (160) for the scalar product,

$$S_m \approx S_\lambda \approx Q(1) \tag{168}$$

and derivatives follow from the identification of multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ from (147)-(152) in section 4.4.1. $Q(F)$ is evaluated from (163) and $Q(1)$. $Q(1)$ does not vary with λ; the multinomials $F(\mathbf{q}_2, \mathbf{q}_4)$ include variation with λ.

4.4.4 Gaussian quadratures

In this section, brief interval, nonrelativistic approximation of the functional $Q(1)$ is evaluated for a range of trajectories determined by $\mathbf{u}, \dot{\mathbf{u}}$. From (163), all $Q(F)$ of interest follow as derivatives of $Q(1)$. For this evaluation, the free field VEV contribution is distinguished from the connected VEV contribution,

$$Q(F) := Q_F(1) + Q_C(1), \tag{169}$$

with

$$Q_F(1) = \frac{a_g}{\lambda_c^2} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2' - \mathbf{p}_4') \; e^{\mathbf{p}_2' \cdot \mathbf{b}_2 + \mathbf{p}_4' \cdot \mathbf{b}_4} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}$$

and

$$Q_C(1) = \frac{a_g c_4}{\lambda_c} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2) \; e^{\mathbf{p}_2' \cdot \mathbf{b}_2 + \mathbf{p}_4' \cdot \mathbf{b}_4} e^{-\overline{\sigma^2}(\mathbf{p}_2' - 2\mathbf{w})^2} \; e^{-\sigma^2(\mathbf{p}_4' - 2\mathbf{w})^2}$$

from the definitions (140) for T_4 and (162) for $Q(F)$.

Evaluation of $Q_F(1)$ and $Q_C(1)$ follows from the Gaussian quadrature

$$\sqrt{\sigma^2} \int_{-\infty}^{\infty} ds \; e^{-\sigma^2 s^2 + \beta s} = \sqrt{\pi} \; e^{\beta^2/(4\sigma^2)} \tag{170}$$

for $\sigma^2, \beta \in \mathbb{C}$ with $\mathbb{R}e(\sigma^2) > 0$.

It is convenient to introduce compact notation for the complex length parameter,

$$\sigma^2 = \sigma_R^2 + i\sigma_Q^2 \tag{171}$$

with $\sigma_R^2, \sigma_Q^2 \in \mathbb{R}$ and $\sigma_R^2 > 0$.

From (169), evaluation of the momentum conservation delta function, $e^{a+b} = e^a e^b$, translation of the summation variables, and substitution of the Gaussian summation (170) result in

$$\begin{aligned} Q_F(1) &= \frac{a_g}{\lambda_c^2} \int d\mathbf{p}_2' \; e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \; e^{-2\sigma_R^2 (\mathbf{p}_2' - 2\mathbf{w})^2} \\ &= \frac{a_g}{\lambda_c^2} e^{2\mathbf{w} \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \int d\mathbf{p}_2' \; e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + \mathbf{b}_4)} \, e^{-2\sigma_R^2 \, \mathbf{p}_2'^2} \\ &= \frac{a_g}{\lambda_c^2} \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{3}{2}} e^{2\mathbf{w} \cdot (\mathbf{b}_2 + \mathbf{b}_4)} e^{\frac{(\mathbf{b}_2 + \mathbf{b}_4)^2}{8\sigma_R^2}} \end{aligned} \tag{172}$$

for the free field VEV contribution to $Q(1)$.

Expressing the summations in spherical coordinates is convenient to evaluate the energy conservation delta function in the connected VEV contribution $Q_C(1)$ to $Q(1)$. Factoring the Gaussian functions in (169) provides that

$$\begin{aligned} Q_C(1) &= \frac{c_4 a_g}{\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int d\mathbf{p}_2' d\mathbf{p}_4' \; \delta(\mathbf{p}_2'^2 - \mathbf{p}_4'^2) \\ &\qquad \times e^{\mathbf{p}_2' \cdot (\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})} e^{\mathbf{p}_4' \cdot (\mathbf{b}_4 + 4\sigma^2 \mathbf{w})} e^{-\overline{\sigma^2} \mathbf{p}_2'^2} e^{-\sigma^2 \mathbf{p}_4'^2}. \end{aligned} \tag{173}$$

A selection of $\mathbf{b}_2 = \overline{\sigma^2}\mathbf{c}_2$, $\mathbf{b}_4 = \sigma^2 \mathbf{c}_4$ with $\mathbf{c}_2, \mathbf{c}_4 \in \mathbb{R}^3$ results in $\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w}$ and $\mathbf{b}_4 + 4\sigma^2\mathbf{w}$ that are complex constants times real spatial vectors. Then, spherical coordinates simplify the summations in (173). Subsequently, the result from evaluation of the energy conserving delta function and summations is analytically extended to evaluate $Q_C(1)$ at the physical values of interest, $\mathbf{b}_2 = -\mathbf{b}_4 = i\mathbf{u}$.

A change of summation variables to spherical coordinates

$$\mathbf{p}_j' = (\rho_j' \cos\theta_j \cos\phi_j, \rho_j' \sin\theta_j \cos\phi_j, \rho_j' \sin\phi_j)$$

with the z_j-axes aligned with the real vectors $\mathbf{c}_j + 4\mathbf{w}$ sets

$$\mathbf{p}_j' \cdot (\mathbf{b}_j + 4\sigma^2 \mathbf{w}) = \rho_j' r_j \sin\phi_j.$$

$j = 2, 4$ and

$$\begin{aligned} r_2 &= \overline{\sigma^2} \left((\mathbf{c}_2 + 4\mathbf{w})^2 \right)^{\frac{1}{2}} \\ &= \left((\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})^2 \right)^{\frac{1}{2}} \\ r_4 &= \sigma^2 \left((\mathbf{c}_4 + 4\mathbf{w})^2 \right)^{\frac{1}{2}} \\ &= \left((\mathbf{b}_4 + 4\sigma^2 \mathbf{w})^2 \right)^{\frac{1}{2}} \end{aligned} \tag{174}$$

with $\mathbb{R}e(r_j) > 0$ from $\mathbb{R}e(\sigma^2) > 0$. Rotational invariance of the state describing function (143) justifies the selection of axes. With the change to spherical coordinates,

$$\begin{aligned} Q_C(1) &= \frac{c_4 a_g}{\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int_0^\infty \rho_2'^2 d\rho_2' \int_0^\infty \rho_4'^2 d\rho_4' \; \delta(\rho_2'^2 - \rho_4'^2) \; e^{-\overline{\sigma^2}\rho_2'^2} e^{-\sigma^2 \rho_4'^2} \\ &\quad \times \int_0^{2\pi} d\theta_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_2 d\phi_2 \; e^{\rho_2' r_2 \sin\phi_2} \int_0^{2\pi} d\theta_4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_4 d\phi_4 \; e^{\rho_4' r_4 \sin\phi_4}. \end{aligned}$$

The θ_j and ϕ_j summations are elementary.

$$\int_0^{2\pi} d\theta_j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi_j \, d\phi_j \; e^{\rho_j' r_j \sin\phi_j} = 2\pi \; \frac{e^{\rho_j' r_j} - e^{-\rho_j' r_j}}{\rho_j' r_j}.$$

The ρ_j' summations are

$$\begin{aligned} \int_0^\infty \rho_j'^2 d\rho_j' \; \frac{e^{\rho_j' r_j} - e^{-\rho_j' r_j}}{\rho_j' r_j} \; h(\rho_j'^2) &= \int_0^\infty \rho_j' d\rho_j' \; \frac{e^{\rho_j' r_j}}{r_j} h(\rho_j'^2) - \int_0^\infty \rho_j' d\rho_j' \; \frac{e^{-\rho_j' r_j}}{r_j} h(\rho_j'^2) \\ &= \int_0^\infty \rho_j' d\rho_j' \; \frac{e^{\rho_j' r_j}}{r_j} h(\rho_j'^2) - \int_0^{-\infty} \rho_j' d\rho_j' \; \frac{e^{\rho_j' r_j}}{r_j} h((-\rho_j')^2) \\ &= \int_{-\infty}^\infty \rho_j' d\rho_j' \; \frac{e^{\rho_j' r_j}}{r_j} \; h(\rho_j'^2) \end{aligned}$$

from reflection of the summation variable in the second term. Denoted by $h(\rho_j'^2)$, both the Gaussian functions and energy conservation delta function are even functions of the ρ_j'. Substitution of these summations into $Q_C(1)$ then provides that

$$Q_C(1) = (2\pi)^2 \frac{c_4 a_g}{\lambda_c} e^{-8\sigma_R^2 \mathbf{w}^2} \int_{-\infty}^\infty \rho_2' d\rho_2' \int_{-\infty}^\infty \rho_4' d\rho_4' \delta(\rho_2'^2 - \rho_4'^2) e^{-2\sigma_R^2 \rho_2'^2} \; \frac{e^{\rho_2' r_2 + \rho_4' r_4}}{r_2 r_4}.$$

The delta function [19] is

$$\delta(\rho_2'^2 - \rho_4'^2) = \frac{\delta(\rho_2' - \rho_4')}{2|\rho_2'|} + \frac{\delta(\rho_2' + \rho_4')}{2|\rho_2'|}$$

and then

$$Q_C(1) = \frac{2\pi^2 c_4 a_g}{\lambda_c r_2 r_4} e^{-8\sigma_R^2 \mathbf{w}^2} \int_{-\infty}^\infty |\rho_2'| d\rho_2' \; e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2'(r_2 + r_4)} - e^{\rho_2'(r_2 - r_4)} \right). \tag{175}$$

Reflecting the summation variable in the domain $(-\infty, 0)$, the remaining summation reorganizes

to

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\rho_2'| d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2'(r_2+r_4)} - e^{\rho_2'(r_2-r_4)} \right) \\
&\quad = \int_0^{\infty} |\rho_2'| d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2'(r_2+r_4)} - e^{\rho_2'(r_2-r_4)} \right) \\
&\qquad - \int_{\infty}^{0} |\rho_2'| d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{-\rho_2'(r_2+r_4)} - e^{-\rho_2'(r_2-r_4)} \right) \\
&\quad = \int_0^{\infty} \rho_2' d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2'(r_2+r_4)} - e^{\rho_2'(r_2-r_4)} + e^{-\rho_2'(r_2+r_4)} - e^{-\rho_2'(r_2-r_4)} \right) \\
&\quad = \int_0^{\infty} \rho_2' d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2' r_2} - e^{-\rho_2' r_2} \right) \left(e^{\rho_2' r_4} - e^{-\rho_2' r_4} \right).
\end{aligned}$$

For large real components of r_2, r_4, a convenient approximation applies. If

$$\mathbb{R}e(r_2), \mathbb{R}e(r_4) \gg 0,$$

then the dominant support of the integrand is for large ρ_2' and

$$e^{\rho_2' r_j} \gg e^{-\rho_2' r_j}.$$

Neglect of the smaller terms and inclusion of the weakly weighted summation over $\rho_2' \in (-\infty, 0)$ approximates (175) when $\mathbb{R}e(r_2), \mathbb{R}e(r_4)$ from (174) are suffiently large. With this approximation,

$$\begin{aligned}
\int_0^{\infty} \rho_2' d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} \left(e^{\rho_2' r_2} - e^{-\rho_2' r_2} \right) \left(e^{\rho_2' r_4} - e^{-\rho_2' r_4} \right) &\approx \int_{-\infty}^{\infty} \rho_2' d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} e^{\rho_2'(r_2+r_4)} \\
&= \frac{\partial}{\partial \mu} \int_{-\infty}^{\infty} d\rho_2' \, e^{-2\sigma_R^2 \rho_2'^2} e^{\rho_2'(r_2+r_4+\mu)} \\
&= \frac{\partial}{\partial \mu} \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{1}{2}} e^{\frac{(r_2+r_4+\mu)^2}{8\sigma_R^2}} \\
&= \left(\frac{\pi}{2\sigma_R^2} \right)^{\frac{1}{2}} \left(\frac{r_2+r_4}{4\sigma_R^2} \right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}}
\end{aligned}$$

from the Gaussian summation (170) and evaluated at $\mu = 0$. Sufficiently large r_2, r_4 is determined to set the peak of the dominant support of $e^{-2\sigma_R^2 \rho_2'^2 + \rho_2'(r_2+r_4)}$ much greater than the width of the support. The envelope of the support is determined by the real component σ_R^2 of σ^2. The width of support is determined from

$$e^{-2\sigma_R^2 \rho_2'^2 + \rho_2' \mathbb{R}e(r_2+r_4)} = e^{-2\sigma_R^2 (\rho_2' - \rho_o)^2} e^{4\sigma_R^2 \rho_o^2}$$

for $\rho_o = \mathbb{R}e(r_2 + r_4)/(4\sigma_R^2)$, the value of ρ_2' at the peak of the envelope of support. Then large r_j is

$$\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\sigma_R^2} \tag{176}$$

for r_2, r_4 that are the analytic extensions of (174). This approximation cannot be arbitrarily accurate with $\sigma_R^2 \to 0$ due to the bound that ensures the support of the state describing function (143) is nonrelativistic.

$$\sigma_R^2 \gg \lambda_c^2$$

from (144). Finally, substitution provides

$$Q_C(1) \approx \frac{2\pi^2 c_4 a_g}{\lambda_c} \left(\frac{\pi}{2\sigma_R^2}\right)^{\frac{1}{2}} \frac{e^{-8\sigma_R^2 \mathbf{w}^2}}{4\sigma_R^2} \left(\frac{r_2 + r_4}{r_2 r_4}\right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}} \tag{177}$$

for $\mathbf{b}_2 = \overline{\sigma^2}\mathbf{c}_2$ and $\mathbf{b}_4 = \sigma^2 \mathbf{c}_4$ with real $\mathbf{c}_j$.

The value of $Q_C(1)$ of physical interest has $\mathbf{b}_j$ from (164). In (177), the $\mathbf{b}_j$ are extended in $\mathbb{C}^3$. After analytic extension,

$$\begin{aligned} r_2 &= \left((i\mathbf{u} + 4\overline{\sigma^2}\mathbf{w})^2\right)^{\frac{1}{2}} \\ r_4 &= \left((-i\mathbf{u} + 4\sigma^2\mathbf{w})^2\right)^{\frac{1}{2}} \end{aligned}$$

and then

$$\overline{r_2} = r_4. \tag{178}$$

Singularities in the analytic extension of the approximation (177) for $Q_C(1)$ include simple divergences at $r_j = 0$ and cut lines due to the multiple sheets of the square root (174) in r_j. Cut lines are oriented toward negative real values of the r_j to avoid the large positive real values of interest. With

$$r_j^2 := a_j + ib_j,$$

$a_j, b_j \in \mathbb{R}$, the root of r_j^2 is selected to set $\mathbb{R}e(r_j) \geq 0$. Then

$$\begin{aligned} \mathbb{R}e(r_j) &= \sqrt{\tfrac{1}{2}\left(\sqrt{a_j^2 + b_j^2} + a_j\right)} \\ \mathbb{I}m(r_j) &= \operatorname{sgn}(b_j)\sqrt{\tfrac{1}{2}\left(\sqrt{a_j^2 + b_j^2} - a_j\right)} \end{aligned} \tag{179}$$

from half-angle formulas for cosine and sine. With the notation (171) for complex $\sigma^2 = L(0)^2$, the physical values of interest follow from

$$\begin{aligned} a_4 &= a_2 \quad &= 16(\sigma_R^4 - \sigma_Q^4)\mathbf{w}^2 - \mathbf{u}^2 + 8\sigma_Q^2 \mathbf{u}\cdot\mathbf{w} \\ b_4 &= -b_2 \quad &= 32\sigma_R^2\sigma_Q^2\mathbf{w}^2 - 8\sigma_R^2 \mathbf{u}\cdot\mathbf{w}. \end{aligned} \tag{180}$$

The analytic extension of the quadrature of interest (173) with $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ equals the summation (175) when

$$\begin{aligned} \mathbf{b}_2 &= \overline{\sigma^2}\mathbf{c}_2 \\ \mathbf{b}_4 &= \sigma^2 \mathbf{c}_4 \end{aligned}$$

and $\mathbf{c}_j \in \mathbb{R}^3$. The summation (175) analytically extends for $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ and within the regions of holomorphy, this extension equals the extension of (173) by the identity theorem. Both (175) and the approximation (177) are functions of r_2, r_4 that, with exclusion of the isolated singularities and cut lines, are functions over $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$. The approximation (177) applies for $\mathbf{b}_2, \mathbf{b}_4 \in \mathbb{C}^3$ with $\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\mathbb{R}e(L(0)^2)}$. If $L(0)^2$, $\mathbf{u}^2$ and $\dot{\mathbf{u}}^2$ satisfy the large r_j condition (176), then the analytic extension of the approximation (177) approximates $Q_C(1)$.

Substitution of the values (164) into (172) and (177) provides the value of interest for $Q(1)$ in the notation (169),

$$\begin{aligned} Q_F(1) &= \frac{4a_g}{\lambda_c^2} \left(\frac{\pi}{2L(0)_R^2} \right)^{\frac{3}{2}} \\ Q_C(1) &= \frac{2\pi^2 c_4 a_g}{4L(0)_R^2 \lambda_c} \left(\frac{\pi}{2L(0)_R^2} \right)^{\frac{1}{2}} e^{-8l(0)_R^2 \mathbf{w}^2} \left(\frac{1}{r_2} + \frac{1}{r_4} \right) e^{\frac{(r_2+r_4)^2}{8L(0)_R^2}} . \end{aligned} \tag{181}$$

The r_j are from (179) with (180). The conditions that produce the approximation of (181) are:

1. nonrelativistic momenta (109), $\dot{\mathbf{u}}^2 \ll 1$, $L(0)_R^2 \gg \lambda_c^2$ and $\lambda_c \mathbf{w} = \dot{\mathbf{u}}$ from (108)
2. significant body separation, $\mathbf{u}^2 \gg L(0)_R^2$, enables the isolation (105) of support to identify a classical body with a region of space and neglect the cross term in the VEV
3. sufficiently brief intervals to neglect $\mathcal{O}(\lambda^2)$ corrections to evolution of the classical dynamical variables (153)
4. limited acceleration (158), $\|\ddot{\mathbf{u}}(0)\| \leq \epsilon \lambda_c / \mathbf{u}^2$, justifies neglect of the envelope evolution correction. Limited error implies a sufficiently limited interval
5. large r_j (176) enables approximation of the scalar products in elementary forms.

4.4.5 Zeros of the derivatives of likelihood $|I(\lambda)|^2$

Following the development in section 4.3, zeros of the derivatives of likelihood $|I(\lambda)|^2$ indicate the most likely corresponding classical trajectory $\mathbf{u}(\lambda)$, (126) and (128). In this section, brief interval, nonrelativistic, limited acceleration, large r_j approximations of the derivatives of the likelihood $|I(\lambda)|^2$ are evaluated. The approximations for the derivatives of likelihood $|I(\lambda)|^2$ are identically zero to $\mathcal{O}(\lambda^2)$ and do not discriminate a most likely trajectory. The nonrelativistic, brief interval, limited acceleration, and large r_j approximations for the zeroes of derivatives of likelihood provide no insight into the corresponding classical trajectories. However, the derivative of the phase of $I(\lambda)$ relates quantum and classical descriptions of the energy and identifies a corresponding classical pair potential. This energy correspondence and the corresponding classical pair potential are developed in section 4.4.6.

The derivatives of $S_m = \langle U(\lambda)\varphi_2(0)|\varphi_2(\lambda)\rangle$ and $S_\lambda = \langle \varphi_2(\lambda)|\varphi_2(\lambda)\rangle$ are developed in section 4.4.1 and the brief interval, nonrelativistic approximations in terms of the functional $Q(F)$ are from section 4.4.4. From (163), $Q(F)$ are derivatives of $Q(1)$. Optimality conditions (130)

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = 0$$

are satisfied if and only if

$$\mathbb{R}e\left(\frac{2}{S_m}\frac{\partial S_m}{\partial \beta}\right) - \frac{1}{S_\lambda}\mathbb{R}e\left(\frac{\partial S_\lambda}{\partial \beta}\right) = 0$$

with β a component of $\mathbf{u}(\lambda)$, a component of $\dot{\mathbf{u}}(\lambda)$ or λ. From (160), the brief interval, nonrelativistic, limited acceleration approximations include

$$S_m \approx S_\lambda \approx Q(1).$$

Expansions of the derivatives of the scalar products in real and imaginary components to first order in λ are designated

$$\begin{aligned} \frac{\partial S_m}{\partial \beta} &:= c_m + id_m + \lambda\alpha_m + i\lambda\epsilon_m \\ \frac{\partial S_\lambda}{\partial \beta} &:= c_\lambda + id_\lambda + \lambda\alpha_\lambda + i\lambda\epsilon_\lambda \end{aligned} \tag{182}$$

with $c_m, d_m, \alpha_m, \epsilon_m \in \mathbb{R}$ and $c_\lambda, d_\lambda, \alpha_\lambda, \epsilon_\lambda \in \mathbb{R}$. In this notation,

$$\begin{aligned} \mathbb{R}e\left(\frac{2}{S_m}\frac{\partial S_m}{\partial \beta}\right) - \frac{1}{S_\lambda}\mathbb{R}e\left(\frac{\partial S_\lambda}{\partial \beta}\right) &= \mathbb{R}e\left(2\,\frac{c_m + id_m + \lambda\alpha_m + i\lambda\epsilon_m}{Q(1)}\right) \\ &\quad -\mathbb{R}e\left(\frac{c_\lambda + id_\lambda + \lambda\alpha_\lambda + i\lambda\epsilon_\lambda}{Q(1)}\right) \\ &= \frac{2c_m - c_\lambda + \lambda(2\alpha_m - \alpha_\lambda)}{Q(1)} \end{aligned}$$

to $\mathcal{O}(\lambda^2)$. $d_m, d_\lambda, \epsilon_m, \epsilon_\lambda$ do not contribute at the zeros of the first derivatives of $|I(\lambda)|^2$. Since $Q(1) \neq 0$, the brief interval optimality conditions are satisfied if

$$2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) = 0 \tag{183}$$

for each selection of β.

Like the brief interval approximation to scalar products (160), substitution of (159) provides that the first derivatives of the scalar products are conveniently approximated with $Q(F)$. With

the abbreviated notations (160), (161) and (162), the first derivatives of S_m with respect to components of $\mathbf{u}(\lambda)$ are from (147),

$$\begin{aligned}\frac{\partial S_m}{\partial \mathbf{u}(\lambda)} &= Q(-i(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})) \\ &= -iQ(\mathbf{q}_4) + 2i\lambda\dot{\mathbf{w}}Q(1).\end{aligned}$$

There is an offset by $2\mathbf{w}$ of the summation variables $\mathbf{p}'_j$ from $\mathbf{q}_j$in the definition (162) of $Q(F)$, and $\mathbf{w}(\lambda)$ is approximated as $\mathbf{w} + \lambda\dot{\mathbf{w}}$. In the designations (182), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned}c_m &= \mathbb{I}m(Q(\mathbf{q}_4)) \\ \alpha_m &= 0.\end{aligned}$$

Derivatives of the squared norm S_λ follow similarly from (148),

$$\begin{aligned}\frac{\partial S_\lambda}{\partial \mathbf{u}(\lambda)} &= Q(i(\mathbf{q}_2 - \mathbf{q}_4)) \\ &= iQ(\mathbf{q}_2 - \mathbf{q}_4).\end{aligned}$$

From transpose symmetry (165) of $Q(F)$,

$$iQ((\mathbf{q}_2 - \mathbf{q}_4) = 2\mathbb{I}m(Q(\mathbf{q}_4))$$

In the designations of (182),

$$\begin{aligned}c_\lambda &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \\ \alpha_\lambda &= 0.\end{aligned}$$

Substitution into (183) for β a component of $\mathbf{u}(\lambda)$ produces

$$\begin{aligned}0 &= 2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) \\ &= 2\mathbb{I}m(Q(\mathbf{q}_4)) - 2\mathbb{I}m(Q(\mathbf{q}_4))\end{aligned}$$

without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

The derivatives of the mixed scalar product S_m with respect to the components of $\dot{\mathbf{u}}(\lambda)$ are from (149) in section 4.4.1.

$$\begin{aligned}\lambda_c\frac{\partial S_m}{\partial \dot{\mathbf{u}}(\lambda)} &= Q(\big(2i(\mathbf{u} + \lambda\dot{\mathbf{u}}) + (4L(0)^2 - i\lambda_c\lambda)(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})\big)) \\ &= Q(\big(2i\mathbf{u} + 4L(0)^2\mathbf{q}_4 + \lambda\left\{2i\dot{\mathbf{u}} - i\lambda_c\mathbf{q}_4 - 8L(0)^2\dot{\mathbf{w}}\right\}\big)) \\ &= 2i\mathbf{u}Q(1) + 4L(0)^2Q(\mathbf{q}_4) + \lambda\left\{(2i\dot{\mathbf{u}} - 8L(0)^2\dot{\mathbf{w}})Q(1) - i\lambda_cQ(\mathbf{q}_4)\right\}\end{aligned}$$

from substitution of (156) for $L(\lambda)^2$, and linear expansion (153) of $\mathbf{u}(\lambda)$ and $\mathbf{w}(\lambda)$. Then, in the notation (182), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned} c_m &= 4\mathbb{R}e(L(0)^2 Q(\mathbf{q}_4)) \\ \alpha_m &= -8\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + \lambda_c \mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

with the common scaling by λ_c^{-1} neglected. The derivatives of the squared norm S_λ are from (150),

$$\begin{aligned} \lambda_c \frac{\partial S_\lambda}{\partial \dot{\mathbf{u}}(\lambda)} &= Q\left((4L(0)^2 + i\lambda_c\lambda)(\mathbf{q}_2 - 2\lambda\dot{\mathbf{w}}) + (4L(0)^2 - i\lambda_c\lambda)(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})\right) \\ &= Q(4L(0)^2\,(\mathbf{q}_2 + \mathbf{q}_4 - 4\lambda\dot{\mathbf{w}}) + i\lambda_c\lambda\,(\mathbf{q}_2 - \mathbf{q}_4)) \\ &= 4L(0)^2 Q(\mathbf{q}_2 + \mathbf{q}_4) + \lambda\left\{-16L(0)^2\dot{\mathbf{w}}Q(1) + i\lambda_c Q(\mathbf{q}_2 - \mathbf{q}_4)\right\} \end{aligned}$$

with substitution of (156) for $L(\lambda)^2$. $Q(\mathbf{q}_2 + \mathbf{q}_4)$ is real from (166) and

$$\begin{aligned} Q(\mathbf{q}_2 + \mathbf{q}_4) &= 2\mathbb{R}e(Q(\mathbf{q}_4)) \\ iQ((\mathbf{q}_2 - \mathbf{q}_4) &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

from (165). In the designations of (182),

$$\begin{aligned} c_\lambda &= 8\mathbb{R}e(L(0)^2 Q(\mathbf{q}_4)) \\ \alpha_\lambda &= -16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + 2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4)) \end{aligned}$$

with the common scaling by λ_c^{-1} neglected.

Substitution into (183) for β a component of $\dot{\mathbf{u}}(\lambda)$ produces

$$\begin{aligned} 0 &= 2c_m - c_\lambda + \lambda\,(2\alpha_m - \alpha_\lambda) \\ &= 8\mathbb{R}e(L(0)^2 Q(\mathbf{q}_4)) - 8\mathbb{R}e(L(0)^2 Q(\mathbf{q}_4)) \\ &\quad + \lambda(-16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) + 2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4)) + 16\mathbb{R}e(L(0)^2)\dot{\mathbf{w}}Q(1) - 2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4))) \end{aligned}$$

also without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

Finally, the partial derivative of the mixed scalar product S_m with respect to λ with $\mathbf{u}(\lambda)$ and $\dot{\mathbf{u}}(\lambda)$ held constant is from (151),

$$\begin{aligned} \frac{\partial S_m}{\partial \lambda} &= Q((2i\lambda_c^{-1} - i\tfrac{\lambda_c}{4}(\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})^2)\,) \\ &= Q((2i\lambda_c^{-1} - i\tfrac{\lambda_c}{4}\mathbf{q}_4^2 + i\lambda\lambda_c(\mathbf{q}_4 \cdot \dot{\mathbf{w}}))) \\ &= 2i\lambda_c^{-1}Q(1) - i\tfrac{\lambda_c}{4}Q(\mathbf{q}_4^2) + i\lambda\left\{\lambda_c Q(\mathbf{q}_4) \cdot \dot{\mathbf{w}}\right\} \end{aligned}$$

from substitution of the expression (156) for $L(\lambda)^2$, and linear expansion (153) of $\mathbf{w}(\lambda)$. Then, in the notation (182), the contributions of S_m to the derivative of likelihood $|I(\lambda)|^2$ follow from

$$\begin{aligned} c_m &= \tfrac{\lambda_c^2}{4}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ \alpha_m &= -\lambda_c \mathbb{I}m(Q(\mathbf{q}_4)) \cdot \dot{\mathbf{w}}. \end{aligned}$$

The squared norm is from (152),

$$\begin{aligned} \frac{\partial S_\lambda}{\partial \lambda} &= Q(i\tfrac{\lambda_c}{4}\left((\mathbf{q}_2 - 2\lambda\dot{\mathbf{w}})^2 - (\mathbf{q}_4 - 2\lambda\dot{\mathbf{w}})^2\right)) \\ &= i\tfrac{\lambda_c}{4}Q(\mathbf{q}_2^2 - \mathbf{q}_4^2) + \lambda\left\{-i\lambda_c Q((\mathbf{q}_2 - \mathbf{q}_4)\cdot\dot{\mathbf{w}})\right\}. \end{aligned}$$

In the designations of (182),

$$\begin{aligned} c_\lambda &= \tfrac{\lambda_c^2}{2}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ \alpha_\lambda &= -2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4)) \cdot \dot{\mathbf{w}}. \end{aligned}$$

$Q(\mathbf{q}_2 - \mathbf{q}_4)$ and $Q(\mathbf{q}_2^2 - \mathbf{q}_4^2)$ are imaginary from (166) and

$$\begin{aligned} iQ((\mathbf{q}_2 - \mathbf{q}_4) &= 2\mathbb{I}m(Q(\mathbf{q}_4)) \\ iQ(\mathbf{q}_2^2 - \mathbf{q}_4^2) &= 2\mathbb{I}m(Q(\mathbf{q}_4^2)) \end{aligned}$$

from (165).

Substitution into (183) for $\beta = \lambda$ produces

$$\begin{aligned} 0 &= 2c_m - c_\lambda + \lambda(2\alpha_m - \alpha_\lambda) \\ &= 2\tfrac{\lambda_c^2}{4}\mathbb{I}m(Q(\mathbf{q}_4^2)) - \tfrac{\lambda_c^2}{2}\mathbb{I}m(Q(\mathbf{q}_4^2)) \\ &\qquad + \lambda(-2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}} + 2\lambda_c \mathbb{I}m(Q(\mathbf{q}_4))\cdot\dot{\mathbf{w}}) \end{aligned}$$

again without constraint on the corresponding trajectory $\mathbf{u}(\lambda)$.

A most likely trajectory $\mathbf{u}(\lambda)$ is not resolved in the nonrelativistic (109), brief interval λ, limited acceleration (158), and large r_j approximations of the derivatives of the likelihood $|I(\lambda)|^2$.

4.4.6 Energy correspondence and $-g/r$ potentials

In this section, the correspondence of quantum and classical expressions for energy is exploited to identify classical trajectories that correspond with the constructed, single neutral scalar field realization of relativistic quantum physics. These classical particle approximations apply when states are described by functions with isolated (105) concentrations of support well represented

by a single location (102) and momentum (103), and the momentum support is nonrelativistic (109).

In section 4.4.5, it is demonstrated that

$$\frac{\partial |I(\lambda)|^2}{\partial \beta} = \mathcal{O}(\lambda^2)$$

for β a component of $\mathbf{u}(\lambda)$, a component of $\dot{\mathbf{u}}(\lambda)$, or λ. The definition (120) of $\phi_I(\lambda)$,

$$|\langle U(\lambda)\varphi_n(0)|\varphi_n(\lambda)\rangle| = \langle U(\lambda)\varphi_n(0)|e^{i\phi_I(\lambda)}\varphi_n(\lambda)\rangle,$$

provides that the temporal derivative of the phase of $I(\lambda)$ and the derivative of $\phi_I(\lambda)$ relate the classical energy from (155) with the energy of the state described by $\varphi_2(0)$.

Evaluated at $\lambda = 0$, with the notation (119) for normalized state describing functions $\varphi_2(\lambda)$, and from (131), the chain rule and $I(0) = 1$, it follows that

$$\begin{aligned} 0 &= \frac{d|I(\lambda)|}{d\lambda} \\ &= i\frac{d}{d\lambda}\langle U(\lambda)\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\hat{\varphi}_2(\lambda)\rangle \\ &= \langle -He^{-iH\lambda}\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\hat{\varphi}_2(\lambda)\rangle - \frac{d\phi_I(\lambda)}{d\lambda}\langle e^{-iH\lambda}\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\hat{\varphi}_2(\lambda)\rangle \\ &\quad + i\langle e^{-iH\lambda}\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\frac{d\hat{\varphi}_2(\lambda)}{d\lambda}\rangle. \end{aligned} \tag{184}$$

For state describing functions of the form (133) with (136) and the classical-like Gaussian descriptions (143), the contribution of the center-of-momentum does not vary with λ and the λ dependence of the description of internals is in the classical dynamical variables $\mathbf{u}(\lambda), \mathbf{w}(\lambda)$ and the parameter $L(\lambda)^2$ (156). From the chain rule, the last term in (184) is recognized as

$$\begin{aligned} \langle e^{-iH\lambda}\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\frac{d\hat{\varphi}_2(\lambda)}{d\lambda}\rangle &= \frac{\partial |I(\lambda)|}{\partial \mathbf{u}(\lambda)}\cdot\dot{\mathbf{u}}(\lambda) + \frac{\partial |I(\lambda)|}{\partial \dot{\mathbf{u}}(\lambda)}\cdot\ddot{\mathbf{u}}(\lambda) + \frac{\partial |I(\lambda)|}{\partial L(\lambda)^2}\frac{dL(\lambda)^2}{d\lambda} \\ &= -i\frac{\partial |I(\lambda)|}{dL(\lambda)^2}\ \frac{\lambda_c}{4} \end{aligned}$$

from the vanishing of derivatives at $\lambda = 0$ developed in section 4.4.5 and with $L(\lambda)^2$ from (156). This contribution vanishes. The contributing terms to

$$\frac{\partial |I(\lambda)|^2}{\partial \lambda} = \mathcal{O}(\lambda^2)$$

in section 4.4.5 include all terms in the partial derivative with respect to $L(\lambda)^2$ except the contribution from the Hamiltonian. Inspection provides that

$$\frac{\partial |I(\lambda)|^2}{\partial L(\lambda)^2}\frac{\lambda_c}{4} = \frac{\partial |I(\lambda)|^2}{\partial \lambda} = 0$$

from the development in section 4.4.5 and the lack of imaginary contributions to $\partial|I(\lambda)|^2/\partial\lambda$ to $\mathcal{O}(\lambda^2)$. Then

$$\langle e^{-iH\lambda}\hat{\varphi}_2(0)|e^{i\phi_I(\lambda)}\frac{d\hat{\varphi}_2(\lambda)}{d\lambda}\rangle = 0$$

and (184) provides that

$$\langle H\varphi_2(0)|\varphi_2(0)\rangle \approx -\frac{d\phi_I(\lambda)}{d\lambda}\langle\varphi_2(0)|\varphi_2(0)\rangle \tag{185}$$

at $\lambda = 0$, with removal of the common normalization factor $\|\varphi_2(0)\|^2$, and from $\phi_I(0) = 0$. From the expression (155) for $\phi_I(\lambda)$,

$$-\frac{d\phi_I(\lambda)}{d\lambda} = (2 + \dot{\mathbf{u}}(0)^2 + 2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0))\frac{1}{\lambda_c}.$$

The expectation value (185) of the energy H equals the classical energy (155). The equality applies for the state describing functions $\varphi_2(0)$ of the form (133) with (136) and (143) in the nonrelativistic, brief interval, limited acceleration approximations.

The development now digresses to evaluates the expectation value of H for the state describing function $\varphi_2(0)$. Substitution of the two-argument state descriptions (137) and VEV (140) into the scalar product $\langle H\varphi_2(0)|\varphi_2(0)\rangle$ provides

$$\begin{aligned}\langle H\varphi_2(0)|\varphi_2(0)\rangle \approx \left(\tfrac{1}{4}\right)^2 \int d\mathbf{p}'_1\ |\tilde{f}_M(\mathbf{p}'_1;\lambda)|^2 \int d\mathbf{p}'_2 d\mathbf{p}'_4\ T_4(\mathbf{p}'_2,\mathbf{p}'_4)\\ \times H\ e^{i(\mathbf{p}'_2-\mathbf{p}'_4)\cdot\mathbf{u}}\ \overline{\tilde{f}_I(\mathbf{p}'_2-2\mathbf{w};0)}\ \tilde{f}_I(\mathbf{p}'_4-2\mathbf{w};0)\end{aligned}$$

from (139) with the abbreviated notation (161) for the initial trajectory parameters. The Hamiltonian (92) in the two-argument subspace is $\omega_1 + \omega_2$. Similarly to the development in section 4.4.1, for nonrelativistic momenta $\lambda_c^2\mathbf{p}_1'^2 \ll 1$,

$$\omega_1 + \omega_2 \approx 2\omega(\frac{1}{2}\mathbf{p}'_2) + \frac{1}{4}\lambda_c\mathbf{p}_1'^2$$

in the Jacobi coordinates (134). A negligible center-of-mass contribution to the energy $\frac{\lambda_c}{4}\mathbf{p}_1'^2$ results if either the center-of-momentum description $\tilde{f}_M$ has a zero expectation or the dominant momentum condition (103) applies. For $\lambda_c^2\mathbf{p}_2'^2 \ll 1$, the approximation (142),

$$2\omega(\frac{\mathbf{p}'_2}{2}) \approx 2\lambda_c^{-1} + \frac{\lambda_c}{4}\mathbf{p}_2'^2,$$

application of $Q(F)$ from (162) and the relation of powers of momenta with derivatives of $Q(1)$ (163) provides

$$\langle H\varphi_2(0)|\varphi_2(0)\rangle \approx \left(2\lambda_c^{-1} + \frac{\lambda_c}{4}\nabla^2_{\mathbf{b}_2}\right)Q(1)$$

in the notation (161) and with a Laplacian

$$\nabla^2_{\mathbf{b}_2} := \sum_\nu \frac{\partial^2}{\partial \mathbf{b}^2_{2\nu}}.$$

The value of physical interest is (164). Subtraction of the rest mass energy from both sides of (185) produces

$$\frac{\lambda_c^2}{4}\nabla^2_{\mathbf{b}_2} Q(1) = (\dot{\mathbf{u}}^2 + 2\ddot{\mathbf{u}}\cdot\mathbf{u})\, Q(1) \tag{186}$$

for the trajectories $\mathbf{u}(\lambda)$ that exhibit a an energy correspondence.

Evaluation of $\nabla^2_{\mathbf{b}_2} Q(1)$ follows from the relationship of powers of momenta with derivatives (163). From identification of free field and connected VEV contributions (169) to $Q(1)$, and their evaluations, (172) and (177) with r_j from (174), $Q(1)$ is a composite function over complex $\mathbf{b}_2, \mathbf{b}_4$.

$$\begin{aligned} Q_F(1) &= \frac{a_g}{\lambda_c^2}\left(\frac{\pi}{2\sigma_R^2}\right)^{\frac{3}{2}} e^{2\mathbf{w}\cdot(\mathbf{b}_2+\mathbf{b}_4)} e^{\frac{(\mathbf{b}_2+\mathbf{b}_4)^2}{8\sigma_R^2}} \\ Q_C(1) &\approx \frac{2\pi^2 c_4 a_g}{\lambda_c}\left(\frac{\pi}{2\sigma_R^2}\right)^{\frac{1}{2}} \frac{e^{-8\sigma_R^2\mathbf{w}^2}}{4\sigma_R^2}\left(\frac{1}{r_2}+\frac{1}{r_4}\right) e^{\frac{(r_2+r_4)^2}{8\sigma_R^2}} \end{aligned}$$

if $\mathbb{R}e(r_2 + r_4) \gg 8\sqrt{\sigma_R^2}$.

To evaluate the gradient of $Q(1)$, it is convenient to designate $\nu \in \{x, y, z\}$, $j \in \{2, 4\}$, and

$$\partial_{j\nu} g(\mathbf{b}_2, \mathbf{b}_4) := \frac{\partial g(\mathbf{b}_2, \mathbf{b}_4)}{\partial \mathbf{b}_{j\nu}}.$$

With $A \in \{F, C\}$, functions $f_{nA}((j\nu)_n)$ are defined

$$\prod_{k=1}^{n} \partial_{j_k\nu_k} Q_A(1) := f_{nA}((j\nu)_n) Q_A(1)$$

with a recursive definition of the functions f_{nA} derived from the product rule for derivatives.

$$\begin{aligned} f_{1A}(j\nu) &:= \frac{\partial_{j\nu} Q_A(1)}{Q_A(1)} \\ f_{k+1,A}((j\nu)_{k+1}) &:= (\partial_{j_{k+1}\nu_{k+1}} f_{kA}((j\nu)_k)) + f_{1A}(j_{k+1}\nu_{k+1}) f_{kA}((j\nu)_k). \end{aligned} \tag{187}$$

The evaluation of $Q_F(1)$ in (172) provides

$$\begin{aligned} f_{1F}(j\nu) &= 2\mathbf{w}_\nu + \frac{\mathbf{b}_{2\nu} + \mathbf{b}_{4\nu}}{4\sigma_R^2} \\ \partial_{j\nu} f_{1F}(j\nu) &= \frac{1}{4\sigma_R^2}. \end{aligned}$$

Then from the definition (187),

$$\begin{aligned}\frac{\lambda_c^2}{4}\nabla^2_{\mathbf{b}_2}Q_F(1) &= \frac{\lambda_c^2}{4}\sum_\nu \left(\partial_{j\nu}f_{1F}(j\nu)+f_{1F}(j\nu)^2\right)Q_F(1)\\ &= \left(\frac{3\lambda_c^2}{16\sigma_R^2}+\dot{\mathbf{u}}^2\right)Q_F(1).\end{aligned}$$

After evaluation of $Q(1)$ at (164), $\mathbf{b}_2=-\mathbf{b}_4$. $\lambda_c\mathbf{w}=\dot{\mathbf{u}}$ in the nonrelativistic approximation.

The evaluation of $Q_C(1)$ in (177) provides

$$\begin{aligned}f_{1C}(j\nu) &= \frac{r_2+r_4}{4\sigma_R^2}\partial_{j\nu}r_j-\frac{(\partial_{j\nu}r_j)}{r_j}\frac{r_{j'}}{r_2+r_4}\\ \partial_{j\nu}f_{1C}(j\nu) &= \frac{r_2+r_4}{4\sigma_R^2}\partial^2_{j\nu}r_j+\frac{(\partial_{j\nu}r_j)^2}{4\sigma_R^2}\\ &\quad -\frac{(\partial^2_{j\nu}r_j)}{r_j}\frac{r_{j'}}{r_2+r_4}+\frac{(\partial_{j\nu}r_j)^2}{r_j^2}\frac{r_{j'}}{r_2+r_4}+\frac{(\partial_{j\nu}r_j)^2}{r_j(r_2+r_4)^2}\,r_{j'}.\end{aligned}$$

with introduction of the notation

$$j' := \begin{cases} 2 & \text{if } j=4\\ 4 & \text{if } j=2.\end{cases}$$

Derivatives of r_j with respect to the components of $\mathbf{b}_\ell$ follow from (174).

$$r_2=\left((\mathbf{b}_2+4\overline{\sigma^2}\mathbf{w})^2\right)^{\frac{1}{2}}$$

and then

$$\begin{aligned}\partial_{2\nu}r_2 &= \frac{\mathbf{b}_{2\nu}+4\overline{\sigma^2}\mathbf{w}_\nu}{r_2}\\ \partial^2_{2\nu}r_2 &= \frac{1}{r_2}-\frac{(\mathbf{b}_{2\nu}+2\overline{\sigma^2}\mathbf{w}_\nu)^2}{r_2^3}.\end{aligned}$$

Then from the definition (187),

$$\begin{aligned}\nabla^2_{\mathbf{b}_2}Q_C(1) &= \left(\sum_\nu \partial_{2\nu}f_{1C}(2\nu)+f_{1C}(2\nu)^2\right)Q_C(1)\\ &\approx \left(\sum_\nu \frac{r_2+r_4}{4\sigma_R^2}\partial^2_{2\nu}r_2+\frac{(\partial_{2\nu}r_2)^2}{4\sigma_R^2}+\left(\frac{r_2+r_4}{4\sigma_R^2}\partial_{2\nu}r_2\right)^2\right)Q_C(1)\\ &\approx \left(\frac{r_2+r_4}{4\sigma_R^2}\right)^2\sum_\nu(\partial_{2\nu}r_2)^2\,Q_C(1)\end{aligned}$$

applying the large r_j approximation to keep the most significant among the proliferation of terms. From (174),

$$\sum_{\nu}(\partial_{2\nu}r_2)^2 = \frac{(\mathbf{b}_2 + 4\overline{\sigma^2}\mathbf{w})^2}{r_2^2} = 1.$$

Collecting results into the quantum-classical energy correspondence (186),

$$\begin{aligned}\langle H\varphi_2(0)|\varphi_2(0)\rangle - 2\lambda_c^{-1}Q(1) &= \frac{\lambda_c^2}{4}\nabla_{\mathbf{b}_2}^2 Q(1)\\ &= \left(\frac{3\lambda_c^2}{16\sigma_R^2} + \dot{\mathbf{u}}^2\right)Q_F(1) + \frac{\lambda_c^2}{4}\left(\frac{r_2+r_4}{4\sigma_R^2}\right)^2 Q_C(1)\\ &= (\dot{\mathbf{u}}^2 + 2\ddot{\mathbf{u}}\cdot\mathbf{u})\,(Q_F(1) + Q_C(1)).\end{aligned} \tag{188}$$

The contributions $Q_F(1)$ and $Q_C(1)$ to $Q(1)$ are from (181). This quantum-classical correspondence (188) relates the expected value of the energy of two elementary particles with an interaction characterized by a single neutral scalar field with the classical energy of two bodies governed by Newtonian mechanics.

(188) exhibits several properties in common with Schrödinger's linear harmonic oscillator example, [49] and section 4. The quantum dynamics determines a corresponding state description, σ^2 in (101) for the linear harmonic oscillator and $L(\lambda)^2$ in the constructed example. There is a correspondence for any classical energy. The state describing functions that exhibit a quantum-classical correspondence do not include energy eigenfunctions of the Hamiltonian that corresponds with the classical dynamics. For the linear harmonic oscillator, observation of the period and amplitude of oscillation for a known mass provide the spring constant k and energy E. σ^2 is determined to satisfy Schródinger's equation for the classical-like Gaussian wave functions. For the constructed realization of relativistic quantum physics, observation of a known mass imply the coupling constant g and the energy. $\mathbb{R}e(L(0)^2)$ is determined by the quantum-classical correspondence (118) for classical-like Gaussian state describing functions (143) analogously to (101) from Schrödinger's study of the linear harmonic oscillator. The potential strength g follows from observation of the corresponding classical trajectories, and $L(0)^2$ is determined from g and the quantum dynamics that includes the coupling constant c_4. $c_4 = 0$ implies that $g = 0$ ($Q_C(1) = 0$). That only particular σ^2 or $\mathbb{R}e(L(0)^2)$ exhibit a quantum-classical correspondence but for any energy is a puzzlement. Another puzzlement is how Schrödinger's equation for nonrelativistic quantum physics derives within quantum mechanics from the realizations of relativistic quantum physics: at the fidelity of the approximations, Schrödinger's equation has the same dynamics as the constructed realization of relativistic quantum physics when nonrelativistic quantum-classical correspondences are exhibited.

The final study within this section verifies that the $2\ddot{\mathbf{u}}\cdot\mathbf{u}$ term in the classical energy is a $-g/r$ pair potential. If the trajectories $\mathbf{u}(\lambda)$ satisfy Newton's equation of motion, then the

$2\ddot{\mathbf{u}}(0)\cdot\mathbf{u}(0)$ term in the $\phi_I(\lambda)$ from (159) is a $-g/r$ pair potential. $\mathbf{u}(\lambda)$ is half the two body separation and Newton's equation for the trajectory $\mathbf{u}(\lambda)$ is then

$$mc^2\ddot{\mathbf{u}}(\lambda) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial(2\mathbf{u}(\lambda))}$$

from (237) in appendix 6.12 and in the units of this note. Identification of the $2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda)$ term as a potential energy and Newton's equation of motion result in

$$\begin{aligned}\frac{V(2\|\mathbf{u}(\lambda)\|)}{mc^2} &= 2\ddot{\mathbf{u}}(\lambda)\cdot\mathbf{u}(\lambda)\\ &= -\frac{1}{mc^2}\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda).\end{aligned} \tag{189}$$

The corresponding potential satisfies

$$V(2\|\mathbf{u}(\lambda)\|) = -\frac{\partial V(2\|\mathbf{u}(\lambda)\|)}{\partial\mathbf{u}(\lambda)}\cdot\mathbf{u}(\lambda).$$

The solution is a $-g/r$ pair potential,

$$V(2\|\mathbf{u}(\lambda)\| = -mc^2\frac{g}{\|2\mathbf{u}(\lambda)\|}.$$

The characteristic length g determines the strength of the potential. The chain rule provides that

$$\begin{aligned}-\left(\nabla_{\mathbf{u}}\frac{1}{\|\mathbf{u}\|}\right)\cdot\mathbf{u} &= -\left(\frac{\partial\|\mathbf{u}\|^{-1}}{\partial\|\mathbf{u}\|}\right)\left(\frac{\partial\|\mathbf{u}\|}{\partial\mathbf{u}_x}\mathbf{u}_x+\frac{\partial\|\mathbf{u}\|}{\partial\mathbf{u}_y}\mathbf{u}_y+\frac{\partial\|\mathbf{u}\|}{\partial\mathbf{u}_z}\mathbf{u}_z\right)\\ &= \frac{1}{\|\mathbf{u}\|^2}\left(\frac{\mathbf{u}_x^2}{\|\mathbf{u}\|}+\frac{\mathbf{u}_y^2}{\|\mathbf{u}\|}+\frac{\mathbf{u}_z^2}{\|\mathbf{u}\|}\right)\\ &= \frac{1}{\|\mathbf{u}\|}.\end{aligned}$$

While approximation of the likelihood results in no insight into the corresponding (118) classical trajectory, the derivatives of $I(\lambda)$ and $\phi_I(\lambda)$ from (155) result in an energy correspondence that identifies the corresponding trajectories $\mathbf{u}(\lambda)$ as solutions of Newton's equation for a $-g/r$ pair potential.

4.5 Extended interval propagation

In this section, satisfaction of the quantum-classical correspondence (118) over brief intervals is extended to longer intervals.

The quantum-classical correspondence (118)

$$U(\ell\lambda)\hat{\varphi}_n(\lambda_o;0) \approx e^{i\phi_I(\ell\lambda)}\hat{\varphi}_n(\lambda_o;\ell\lambda)$$

is expressed as a sequence of brief interval approximations,

$$U(\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) \approx \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o;\ell\lambda)$$

using the group property of time translation and additivity of the phase $\phi_I(\lambda)$. Here, refering to section 4.3, the notation is augmented to explicitly display the time argument λ_o of the state describing function as well as the temporal parameter for the corresponding classical trajectories.

$$U(\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) = \hat{\varphi}_n(\lambda_o-\ell\lambda;0)$$

from (32). The error in the quantum-classical correspondence (118) for an interval $\ell\lambda$ follows from the approximation errors for each subinterval within $\ell\lambda$. The error at step ℓ is described by state describing functions ϵ_ℓ.

$$U(\lambda)\hat{\varphi}_n(\lambda_o-\ell\lambda;\ell\lambda) := e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o-\ell\lambda;(\ell+1)\lambda)+\epsilon_{\ell+1}. \tag{190}$$

The error after accumulation of the ℓ steps of duration λ is described by ϵ.

$$U(\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) := \left(e^{i\phi_I(\lambda)}\right)^\ell \hat{\varphi}_n(\lambda_o;n\lambda)+\epsilon.$$

From successive substitution of (190) it follows that

$$\begin{aligned}
U(\lambda)^\ell\hat{\varphi}_n(\lambda_o;0) &= U(\lambda)^{\ell-1}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o;\lambda)+\epsilon_1\right)\\
&= e^{i\phi_I(\lambda)}U(\lambda)^{\ell-1}\hat{\varphi}_n(\lambda_o;\lambda)+U(\lambda)^{\ell-1}\epsilon_1\\
&= e^{i\phi_I(\lambda)}U(\lambda)^{\ell-2}\hat{\varphi}_n(\lambda_o-\lambda;\lambda)+U(\lambda)^{\ell-1}\epsilon_1\\
&= e^{i\phi_I(\lambda)}U(\lambda)^{\ell-3}\left(e^{i\phi_I(\lambda)}\hat{\varphi}_n(\lambda_o-\lambda;2\lambda)+\epsilon_2\right)+U(\lambda)^{\ell-1}\epsilon_1\\
&= \left(e^{i\phi_I(\lambda)}\right)^2U(\lambda)^{\ell-4}\hat{\varphi}_n(\lambda_o-2\lambda;2\lambda)+e^{i\phi_I(\lambda)}U(\lambda)^{\ell-3}\epsilon_2+U(\lambda)^{\ell-1}\epsilon_1\\
&= \ddots\\
&= \left(e^{i\phi_I(\lambda)}\right)^\ell U(\lambda)^{-\ell}\hat{\varphi}_n(\lambda_o-\ell\lambda;\ell\lambda)+\sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1}U(\lambda)^{\ell+1-2j}\epsilon_j\\
&= \left(e^{i\phi_I(\lambda)}\right)^\ell\hat{\varphi}_n(\lambda_o;\ell\lambda)+\sum_{j=1}^{\ell}\left(e^{i\phi_I(\lambda)}\right)^{j-1}U(\lambda)^{\ell+1-2j}\epsilon_j.
\end{aligned}$$

Then, the error ϵ after ℓ steps is

$$\epsilon = \sum_{j=1}^{\ell} \left(e^{i\phi_I(\lambda)} \right)^{j-1} U(\lambda)^{\ell+1-2j} \epsilon_j$$

from the errors ϵ_j for each subinterval. From the triangle inequality, and unitarity of $U(\lambda)$ and $e^{i\phi_I(\lambda)}$,

$$\|\epsilon\| \leq \sum_{j=1}^{\ell} \|\epsilon_j\|.$$

The accumulated error grows no faster than the number of subintervals. If the errors $\|\epsilon_j\|$ decline faster than the duration of the subintervals, then the limit of an ever briefer short interval analysis converges and demonstrates a correspondence over an extended interval. For instance, if the brief intervals errors were all $\mathcal{O}(\lambda^2)$, then the error over a fixed interval would converge as $1/\ell$. However, the large r_j approximation does not improve with the duration of the subinterval, and the nonrelativistic Hamiltonian and limited acceleration approximations improve no better that linearly with the number of subintervals. Extension of the interval applies only as long C.1-3 remain valid.

Brief interval propagation is also of interest for recurring observations. Recurring observations, for example, massive bodies awash in photons, are common. Then, an observed trajectory results from the accumulation of brief interval likelihoods in a random walk composed of the likely trajectories.

5 Technical concerns with canonical quantization

Inquiry into an appropriate mathematical development of quantum mechanics was initiated notably by John von Neumann. This inquiry has been extended by Léon van Hove, Res Jost, Rudolf Haag, Arthur Wightman, Huzihiro Araki, Nikolay Bogolubov, Hans-Jürgen Borchers, Raphael Høegh-Krohn, Franco Strocchi and many others [9, 67]. Their developments use the concept of Hilbert space advanced notably by David Hilbert, Erhard Schmidt, Frigyes Riesz, Marshall Stone and John von Neumann. The discussion here distinguishes a general development of quantum mechanics from a canonical formalism development. The general development includes: quantum mechanics describes the states of nature as elements of rigged Hilbert spaces; the evolution of the observable features of appropriate states is well approximated by classical mechanics; energies are nonnegative; the temporal evolution of state descriptions is unitary (likelihood preserving) and causal; likelihoods of observation are calculated from Born's rule; and likelihoods, like events, are relativistically invariant. The canonical formalism adds the conjecture that the quantum-classical correspondence is established be associating classical dynamical variables with densely defined Hermitian operators [13, 57, 60]. Concerns with the

canonical formalism are introduced in section 2. This section extends the discussion of technical concerns with the canonical formalism for quantum mechanics.

Quantum mechanics has provided an expanded and successful description of nature. Nevertheless, and emphasized by von Neumann, issues remain even in the development of nonrelativistic quantum mechanics [60]. The canonical formalism [13, 57, 61] "quantizes" classical dynamics. If classical dynamical variables quantize to Hermitian Hilbert space operators and there is a "quantization" of the corresponding classical dynamics, is it a concern that while functions of classical dynamical variables are classical dynamical variables (for example, generalized coordinates), products of Hermitian operators are not necessarily Hermitian operators? Are some classical quantities distinguished by having quantizations while other quantities do not? Or is it a concern that the (generalized) eigenfunctions of operators associated with observables by the canonical formalism are not always elements of the Hilbert space, that is, do not describe states in nature? In such cases, what state results from a "collapse to an eigenstate of the observable?" The Schrödinger representation of location X_ν and momentum P_ν operators satisfy the Born-Heisenberg-Jordan relation $[X_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$ and serve as archetypes for a quantization of classical dynamical variables. These X_ν and P_ν apply in the $\mathcal{L}^2$ Hilbert spaces suitable for nonrelativistic quantum mechanics. In the following few paragraphs, the discussion includes that: the eigenfunctions of the operators X_ν and P_ν are not elements of $\mathcal{L}^2$ Hilbert spaces; locations and momenta associated with state describing functions are well defined as observable features even when expectations of the corresponding operators X_ν and P_ν diverge; the quantization of location X_ν fails to be a Hermitian operator even in relativistic free field theory; and the quantization of products of locations $\mathbf{x}$ and momenta $\mathbf{p}$ are not necessarily Hermitian operators even in nonrelativistic quantum mechanics. Each of these points is a difficulty or ambiguity for canonical quantization. Together contradictions motivate a revised quantum-classical correspondence.

First, greater detail on Hilbert space operators is introduced. Discussion is limited to the complex Hilbert spaces of interest. Study of Hilbert space operators, particularly unbounded operators in infinite dimensional Hilbert spaces, is a subtle and elegant subject [40] that illustrates many of the "paradoxes of infinity." In appendix 6.2.5, a Hilbert space operator A is introduced as a mapping of elements from a Hilbert space $\mathbf{H}$ back into $\mathbf{H}$, $A : \mathbf{H} \mapsto \mathbf{H}$. If

$$|A\psi\rangle = |g\rangle$$

then $|\psi\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$, the domain of A, and $|g\rangle \in \mathcal{R}_A \subseteq \mathbf{H}$, the range of A. The domains of bounded operators can be extended to the entire $\mathbf{H}$. A is bounded if $\|A\psi\| \leq c\|\psi\|$ for some $c \in \mathbb{R}$ independently of the element $|\psi\rangle$. The least upper bound c defines an operator norm $\|A\|$. The domain of an unbounded A is necessarily a proper subset of the Hilbert space but the domain may be dense in $\mathbf{H}$. A set of elements $|e_n\rangle$ is *dense* in $\mathbf{H}$ if every $|\psi\rangle \in \mathbf{H}$ is within

an arbitrarily small neighborhood of a finite linear combination of the $|e_n\rangle$,

$$\|\psi - \sum_{n=1}^{N} c_n e_n\| < \epsilon$$

for $N > N_\epsilon$ with the $c_n \in \mathbb{C}$ and $\epsilon \to 0$ as N grows. A *separable* Hilbert space has a denumerable (finite or infinite) dense set of elements $|e_n\rangle$. An example is that functions in the Schwartz function space $\underline{\mathcal{S}}$ [20] are dense in the square-summable functions in $\mathcal{L}^2$. $\mathcal{L}^2$ has a dense, denumerable basis. For an unbounded operator B, a subsequence from a diverging sequence $|Bu_n\rangle$ can be selected and relabeled such that $\|Bu_n\| > n\|u_n\|$. Then, the neighborhood of each element $|\psi\rangle$ with $|B\psi\rangle \in \mathbf{H}$ contains a sequence constructed below with $\|B(\psi - \psi_n)\| > n\|\psi - \psi_n\|$ for n growing without bound and $\|\psi - \psi_n\| \to 0$. That is, an unbounded operator is not continuous anywhere in the Hilbert space. Indeed, using the divergent sequence $|u_n\rangle$ selected above, set

$$|v_n\rangle := \frac{|u_n\rangle}{n\|u_n\|}, \quad \text{and then} \qquad \|Bv_n\| = \frac{\|Bu_n\|}{n\|u_n\|} > 1.$$

The sequence $|\psi_n\rangle := |\psi\rangle + |v_n\rangle$ is a convergent sequence, $\|\psi - \psi_n\| = \|v_n\| = 1/n \to 0$, but $\|B(\psi - \psi_n)\| = \|Bv_n\| > 1$. Then generally, neighboring states within a Hilbert space do not have nearly the same observables when observables are associated with unbounded Hilbert space operators.

In contrast, states exhibit observable features even when the expectation values of corresponding operators diverge or when the operation is undefined. For the example of location,

$$\psi(x) = e^{-\mathbf{x}^2/(4\sigma^2)} + \frac{\epsilon}{(1+\mathbf{x}^2)^{1-\delta}}$$

is dominantly supported near $\mathbf{x} = 0$ for $0 < \epsilon, \delta \ll 1$ with the length σ determining the size of the neighborhood. The σ, ϵ, δ can be selected to provide an arbitrarily large likelihood that an observation of location is near $\mathbf{x} = 0$. For the dominant support, the likelihood that the perceived location will be within a finite volume of dominant support is much greater that the likelihood the perceived location will be from any other disjoint, equal volume. In this interpretation, the likely location is near $\mathbf{x} = 0$ for the state described by $\psi(x)$ even though $\langle\psi|\mathbf{x}\psi\rangle$ diverges. For location, $x_\nu \mapsto X_\nu$ is the Hilbert space location operator, and the expectation value $\langle\psi|X_\nu\psi\rangle$ diverges even though the likely perception of the body described by $\psi(x)$ is that the body will be observed near the origin. The likelihod contrast is made for obervations over finite volumes. At large $\mathbf{x}^2$ (the unobservable "far side of the moon"), there is a small likelihood of detection within any finite volume but a sufficient volume for the mean value to diverge. There is no actual divergence since we are not capable of detecting location over infinite volumes. Our measurements are always localized; location X_ν and momentum P_ν are idealized observables. Actual observations are limited to finite, localized volumes: without an omniscient classical

observer, no knowledge of infinite volumes is available. The likelihoods of physical interest are relative likelihoods of detection comparing finite volumes. Unknowable, distant support of functions only hypothetically affects our considerations. In the example, $\|\psi(x) - e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is small, they are neighboring states, but $\|X_\nu\psi(x) - X_\nu e^{-\mathbf{x}^2/(4\sigma^2)}\|$ is divergent in the $\mathcal{L}^2$ norm.

Correspondence with the classical concepts of location and momentum is provided by the support of the functions that describe states. Considering support, the likelihoods of observations of location and momenta are nearly the same if the support of the state describing functions are nearly the same. The support of state describing functions are more robust and consistent observable features than expected values of unbounded, densely defined Hermitian Hilbert space operators like X_ν and P_ν.

The observables, location, momentum and field strength, are anticipated to have eigenvalues for every real number. As a consequence, their eigenfunctions cannot be elements of $\mathbf{H}_\mathcal{P}$. At best, their eigenfunctions are generalized eigenfunctions, [21] and appendix 6.2.1. Eigenfunctions with distinct eigenvalues of a Hermitian operator are orthogonal [46] and there can only be a countable number of mutually orthogonal functions within the separable Hilbert spaces of interest [10]. Observation of location, momentum and field strength can not be "collapse" to an eigenfunction of the observable: generalized eigenfunctions are not admissible state describing functions.

Introduced in section 2, extrapolation of the quantized location operator X_ν to relativistic physics does not succeed. With consideration of relativity, the X_ν that results from quantization of the classical dynamical variables x_ν are not Hermitian operators. As illustrated in appendix 6.2.6, operators whose eigenvalues are observable quantities must be Hermitian. Transition likelihoods are determined by the scalar product and these likelihoods describe events independently of inertial observer. As a consequence, the scalar product must be a Lorentz invariant in relativistic physics. The scalar product (21), and the Källén-Lehmann form for the two-point function as a nonnegative summation over masses of the Pauli-Jordan function result in consideration of

$$\begin{aligned}\langle X_\nu^*\psi|g\rangle &= \langle\psi|X_\nu g\rangle \\ &= \int dxdy\ \Delta^+(y-x)\,\overline{\psi(y)}\,x_\nu g(x) \\ &\neq \langle X_\nu\psi|g\rangle \\ &= \int dxdy\ \Delta^+(y-x)\,y_\nu\overline{\psi(y)}\,g(x)\end{aligned}$$

since the Pauli-Jordan function is not of point support in spacetime, $x_\nu\Delta^+(y-x) \neq y_\nu\Delta^+(y-x)$. Then $X_\nu^* \neq X_\nu$ for $\psi, g \in \mathcal{D}_{X_\nu}$. The elevation of $\mathbf{x}_\nu$ to Hilbert space operator X_ν is not Hermitian in relativistic quantum mechanics. This "localization problem" is one of many problems

that develop with the canonical formalism. In contrast, the energy-momentum operators,

$$\sum_{\nu=1}^{n} P_\nu$$

in the n-argument subspace, are generators of translations in a unitary realization of the translation group and are self-adjoint as a consequence of Stone's theorem [24, 40]. In the argument above, $\langle\psi|P_\nu g\rangle = \langle P_\nu\psi|g\rangle$ follows from the point support of the Källén-Lehmann form in the momentum domain,

$$p_\nu\delta(p-q) = \delta(p-q)q_\mu.$$

A unitary realization of the translation group results from translation invariance of the scalar product. Location, an observable of evident importance in physics, does not correspond precisely to the elevation of $\mathbf{x}_\nu$ in relativistic quantum mechanics. Discussed in section 2 and appendix 6.4, the Hermitian operator associated with location is not the quantization of x_ν. Suitable location operators are determined by the relativistically invariant localized functions that describe those states that most closely correspond to a body at a particular location in relativistic quantum physics. These forms [43] are discussed in appendix 6.3.

Quantizations of classical dynamical variables can be excluded as Hermitian operators even in ordinary (nonrelativistic) quantum mechanics. The product x^3p can be considered a classical dynamical quantity and in ordinary quantum mechanics with a single spatial dimension, the corresponding operator in $\mathcal{L}^2$ should be the formally Hermitian

$$i\hbar x^{3/2}\frac{d}{dx}x^{3/2} = i\hbar(\frac{x^3}{2}\frac{d}{dx} + \frac{d}{dx}\frac{x^3}{2})$$

since the classical dynamic variables x and p correspond to unbounded, self-adjoint operators x and $i\hbar d/dx$ in this one spatial dimensional $\mathcal{L}^2$ example. However, this densely defined, formally Hermitian operator that corresponds with x^3p has square-summable eigenfunctions

$$s_\lambda(x) := \sqrt{2\lambda}\ \frac{\exp\left(-\lambda/(2x^2)\right)}{x^{3/2}}$$

with imaginary eigenvalues $-i\hbar\lambda$ [9]. $0 < \lambda \in \mathbb{R}$. As a consequence, $X^{3/2}PX^{3/2}$ is not a densely defined Hermitian operator in the $\mathcal{L}^2$ Hilbert space. This $s_\lambda(x) \in \mathcal{L}^2$ is defined for $x > 0$ and equals zero otherwise, or $s_\lambda(x)$ can be extended to negative x. The formally Hermitian operator $X^{3/2}PX^{3/2}$ that corresponds to x^3p is not Hermitian for $\mathcal{L}^2$ although x^3p is well defined in classical dynamics. Nevertheless, for the example of linear harmonic motion and minimum uncertainty support states $s_t(x)$ with small spatial variances, the trajectory of x^3p given by Newtonian mechanics approximates $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ from quantum dynamics [34, 49]. This establishes that there are particular states with real $\langle s_t|X^{3/2}PX^{3/2}s_t\rangle$ that agree with the classical approximations x^3p even though $X^{3/2}PX^{3/2}$ is not a Hermitian operator.

The Hermitian operator corresponding to x^3p need not be the elevation of x^3p. Once again, the support of functions for appropriate state describing functions exhibit quantum-classical correspondence while a correspondence of classical dynamical variable with an operator fails. Even when the operator that is the quantization of the classical dynamical variable is not Hermitian, the classical dynamics can approximate the quantum mechanics for appropriate states.

6 Appendices

6.1 The Dirac-von Neumann axioms for quantum mechanics

The Dirac-von Neumann axioms describe nonrelativistic quantum mechanics. Here, the statement of the Dirac-von Neumann axioms is adapted from F. Strocchi's discussion [57].

I. *Observables*: The Hermitian operators A corresponding to the observables of a quantum mechanical system are within the algebra of bounded self-adjoint operators $\mathcal{B}(\mathbf{H})$ for a separable Hilbert space $\mathbf{H}$.

II. *States*: The pure states of a quantum mechanical system are described by rays $a|\underline{s}\rangle$, $|s\rangle \in \mathbf{H}$, $a \in \mathbb{C}$ and $a \neq 0$. More generally, a state is described by a nonnegative, unit trace, state density operator $\rho \in \mathcal{B}(\mathbf{H})$.

III. *Expectations*: If a state is represented by the normalized pure state $|s\rangle \in \mathbf{H}$, then, for the observable corresponding to $A \in \mathcal{B}(\mathbf{H})$, the experimental expectation is $\langle s|As\rangle$. More generally, experimental expectations are Trace$(A\rho)$. If A has a complete set of normalized eigenvectors $|e_n\rangle \in \mathbf{H}$, then

$$\langle s|As\rangle = \sum_n \lambda_n |\langle e_n|s\rangle|^2$$

and from Born's rule, the likelihood of observing λ_n is $|\langle e_n|s\rangle|^2$. More generally, A is described by the spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [40], chapters 7-10 [24]).

Axioms I-III describe a Hilbert space realization of quantum mechanics and are implicit in the development of section 3. Conditions A.1 and A.2 of section 3.2 imply a Hilbert space realization. State density operators $\rho \in \mathcal{B}(\mathbf{H})$ are discussed in section 6.2.6. States are described by elements of the Hilbert space: limits of states such as the eigenstates of location and momentum are in the dual to the basis function spaces but depart from the rigged Hilbert spaces of interest [9]. Superselection sectors in the Hilbert spaces of interest illustrate that not all Hilbert space elements represent states of nature. Hilbert spaces of interest are represented as direct sums of superselection sectors and the observables are associated with operators limited to within sectors. The limitation of observables to bounded operators departs from Dirac's development but

reflects that only finite values are observable, for example, locations within an accessible, finite volume. Limiting consideration to bounded operators provides several technical conveniences [24, 57] as well as conceding to physical reality.

Axioms I-III describe a general Hilbert space realization of quantum mechanics. Characterization of the observables remains. Dirac-von Neumann axioms IV-V determine properties of particular operators.

IV. *Dirac canonical quantization*: The Hermitian operators that describe canonical coordinates q_i and momenta p_j , $i, j = 1, \ldots N$, obey canonical commutation relations

$$\begin{aligned} [q_i, q_j] &= 0 \\ [p_i, p_j] &= 0 \\ [q_i, p_j] &= -i\hbar\delta_{ij}. \end{aligned}$$

V. *Schrödinger representation*: The canonical commutation relations are realized in the Hilbert space

$$\mathbf{H} := \mathcal{L}^2 = \{\psi(x) | \int d\mathbf{x}\, |\psi(ct, \mathbf{x})|^2 < \infty, x \in \mathbb{R}^4, \mathbf{x} = x_1, x_2, x_3\}$$

by:

$$\begin{aligned} |q_i\psi\rangle &:= x_i\, |\psi\rangle \\ |p_i\psi\rangle &:= -i\hbar\, |\frac{\partial\psi}{\partial x_i}\rangle. \end{aligned}$$

In the Dirac-von Neumann axioms, canonical variables q_i, p_i are quantizations of classical dynamical variables. If the canonical variables q_i, p_i are location and momentum, respectively, the canonical commutation relations are known as the Born-Heisenberg-Jordan relations. In the $\mathcal{L}^2$ Hilbert space applicable to nonrelativistic physics, these quantizations result in unbounded Hermitian Hilbert space operators q_i, p_i, violating I. The correspondence in axiom V is the "elevation of c-number to q-number" for q_i. Unboundedness introduces the consideration that sums of observables are not necessarily observables (if necessarily limited domains are sufficiently disjoint) [24]. And, the (generalized) eigenstates of location and momentum are not elements of the Hilbert space. These and additional difficulties with the canonical formalism are discussed in section 5. Axiom V describes a particular, nonrelativistic quantum-classical correspondence for location and momentum. The canonical formalism extrapolates axioms IV-V with fields as canonical coordinates [61]. Axiom IV remains valid for location and momentum in the constructions.

> The lack of a clear distinction between the role of the two sets of axioms, I, II, III and IV,V, is at the origin of the widespread point of view, adopted by many textbooks, by which all of them are characteristic of quantum systems. The distinction between classical and quantum systems is rather given by the mathematical structure of [the algebra of observables] and it will have different realizations depending on the particular class of systems. – Franco Strocchi [57].

6.2 Hilbert spaces and quantum mechanics

6.2.1 Hilbert spaces

A Hilbert space **H** is characterized by the number of linearly independent elements. For every two elements in the complex Hilbert spaces of interest here, there is a complex number $\langle \underline{f}|\underline{g}\rangle$,

$$\underline{f}, \underline{g} \in \mathbf{H} \mapsto \langle \underline{f}|\underline{g}\rangle \in \mathbb{C}$$

designated the *scalar product* of the elements. Properties of this scalar product include that $\langle \underline{g}|\underline{f}\rangle$ is the complex conjugate of $\langle \underline{f}|\underline{g}\rangle$ and the scalar product is linear in the second argument, $\langle \underline{f}|\alpha\underline{g} + \beta\underline{h}\rangle = \alpha\langle \underline{f}|\underline{g}\rangle + \beta\langle \underline{f}|\underline{h}\rangle$ for $\alpha, \beta \in \mathbb{C}$, and as a consequence, complex conjugate linear in the first argument. Scalar products are nonnegative, $\langle \underline{f}|\underline{f}\rangle \geq 0$, and this provides satisfaction of the Cauchy-Schwarz-Bunyakovsky inequality,

$$|\langle \underline{g}|\underline{f}\rangle|^2 \leq \langle \underline{g}|\underline{g}\rangle\langle \underline{f}|\underline{f}\rangle.$$

In particular, if $\langle \underline{f}|\underline{f}\rangle = 0$, then $\langle \underline{g}|\underline{f}\rangle = 0$ for every element $|\underline{g}\rangle$ of the Hilbert space. The zero element is unique in a Hilbert space. A *degenerate scalar product* has all the properties of a scalar product except for uniqueness of the zero element. A degenerate scalar product (21) is defined for pairs of function sequences from the basis spaces $\underline{\mathcal{P}}$ considered as a linear vector space. The Hilbert spaces of interest are the completions of linear vector spaces with elements that are equivalence classes of vectors labeled by function sequences [12]. The elements of the Hilbert space may be characterized by any function sequence in an equivalence class. An isometry extends the degenerate scalar product to a scalar product: the elements of the Hilbert space are equivalence classes of functions equal in the norm (20) and $\langle \underline{f}|\underline{f}\rangle = 0$ states that the function sequence $\underline{f}$ is an element of the equivalence class of zero. The separation of two Hilbert space elements $\underline{g}, \underline{f}$ is $\|\underline{g} - \underline{f}\|$ using the norm (20). The distance is zero for two equivalent function sequences, and two is the maximum separation of *normalized* state descriptions ($\|\underline{f}\| = \|\underline{g}\| = 1$). The separation is $\sqrt{2}$ for orthogonal, normalized states. Two states $\underline{f}, \underline{g}$ are *orthogonal* if $\langle \underline{g}|\underline{f}\rangle = 0$. In quantum mechanics, every element on a ray $|a\underline{f}\rangle$ with finite $a \in \mathbb{C}$ describes the same physical state: only relative phase and amplitude within linear combinations are significant. Linearity and completeness are characteristic of Hilbert spaces. A Hilbert space **H** is complete: the limit of every Cauchy sequence of elements $|\underline{g}_\nu\rangle \in \mathbf{H}$ is included. That is, if

$$\|\underline{g}_\nu - \underline{g}_n\| \to 0$$

for $\nu, n > N \to \infty$, there is an element $|\underline{g}\rangle \in \mathbf{H}$ such that $|\underline{g}\rangle = \lim_{\nu\to\infty} |\underline{g}_\nu\rangle$. In a *separable* Hilbert space, every element is arbitrarily well approximated by a denumerably indexed sum of N linearly independent elements.

$$|\underline{f}\rangle = \sum_\ell c_\ell |e_\ell\rangle$$

with $\ell \in \mathbb{N}$, the natural numbers, and $c_\ell \in \mathbb{C}$. A (closed) subspace of a Hilbert space is the linear span of a subset of elements. If there are only N linearly independent elements, the Hilbert space or subspace is finite dimensional of dimension N, and if the number of linearly independent elements is unbounded but includes a denumerable, dense set of elements, the Hilbert space or subspace is denoted infinite dimensional and separable. The rigged Hilbert spaces of interest here are separable [10].

6.2.2 Rigged Hilbert spaces

Rigged (equipped) Hilbert spaces are appropriate settings for quantum mechanics. VEV that provide a Poincaré invariant scalar product (21) must be generalized functions: the VEV can not be summable functions [9]. The basis function spaces $\underline{\mathcal{P}}$ used in the constructions include only those elements from $\underline{\mathcal{S}}$ with Fourier transforms that vanish on the negative energy mass shells. Three classes of functions, denoted a Gelfand triple after Israel Gelfand, describe a rigged Hilbert space. The elements of a countably normed basis function space are denoted *test functions* [9, 19, 20]. A particularly useful space of test functions are the Schwartz tempered functions $\underline{\mathcal{S}}$: the tempered functions are smooth (infinitely differentiable) and exhibit rapid decline for large values of their arguments [20]. The space of Fourier transforms of $\underline{\mathcal{S}}$ coincides with $\underline{\mathcal{S}}$. The associated class of generalized functions $\underline{\mathcal{S}}'$ are the linear distributions (generalized functions) dual to $\underline{\mathcal{S}}$. Linear distributions $T(x)$ map functions to complex numbers.

$$T(x) \in \underline{\mathcal{S}}' : \psi(x) \in \underline{\mathcal{S}} \mapsto T(\psi) \in \mathbb{C}.$$

$\underline{\mathcal{S}}'$ is usefully conceived as limits of functions $T(x)$ such that

$$T(\psi) = \int dx\, T(x)\psi(x)$$

is finite when $\psi(x) \in \underline{\mathcal{S}}$ with acknowledgment that this concept includes limits that are not summable using Lebesgue measure.[n] Indeed, the generalized functions dual to the functions of bounded support can be represented

$$T(\psi) = \sum_n \int d\mu_k(x)\, \frac{d^n\psi(x)}{dx^n}$$

[n] A perspective on the distinction between functions and generalized functions is illustrated by generalized functions with a single point of support. Both $\delta(x)$ and $\delta'(x)$ are limits of test functions and supported solely on the point $x = 0$ but $\delta(f) = f(0)$ and $\delta'(f) = f'(0)$.

with the summation over a finite number of terms using measures μ_k on the real numbers [20]. $\mathbf{H}_{\mathcal{S}}$ consists of $\underline{\mathcal{S}}$ plus the Cauchy sequences convergent in a Hilbert space norm (20). As a consequence, the Gelfand triple $(\mathcal{S}, \mathbf{H}_{\mathcal{S}}, \mathcal{S}')$ satisfies

$$\underline{\mathcal{S}} \subset \mathbf{H}_{\mathcal{S}} \subset \underline{\mathcal{S}}'.$$

The eigenfunctions of location and momentum are not elements of $\mathbf{H}_{\mathcal{S}}$ but are elements of $\underline{\mathcal{S}}'$. *Generalized eigenfunctions* of a linear operator A defined in $\underline{\mathcal{S}}$ are the generalized functions $T_\lambda(x) \in \underline{\mathcal{S}}'$ such that $T_\lambda(Af) = \lambda T_\lambda(f)$ for any $f \in \mathcal{S}$.

The function spaces $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are a union of nuclear, countably normed spaces [21]. The elements of $\underline{\mathcal{P}} \subset \underline{\mathcal{S}}$ are test functions and their limits within $\mathbf{H}_{\mathcal{P}}$ constructed from the VEV in section 3 include generalized functions.

6.2.3 The Cauchy-Schwarz-Bunyakovsky inequality

The Cauchy-Schwarz-Bunyakovsky [46] inequality is that

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$$

for elements u, v within a complex linear vector space $\mathbf{H}$. The Cauchy-Schwarz-Bunyakovsky inequality applies if the vector space has a product $\langle u|v\rangle : u, v \in \mathbf{H} \mapsto \langle u|v\rangle \in \mathbb{C}$ with properties

$$\begin{aligned} \langle v|v\rangle &\geq 0 \\ \langle u|v\rangle &= \overline{\langle v|u\rangle} \\ \langle w|\alpha u + \beta v\rangle &= \alpha\langle w|u\rangle + \beta\langle w|v\rangle \end{aligned} \tag{191}$$

for all u, v, w in the vector space $\mathbf{H}$ and $\alpha, \beta \in \mathbb{C}$. This product is designated a complex scalar product if $\langle v|v\rangle = 0$ implies $v = 0$, and otherwise it is a degenerate scalar product.

For any two elements in the vector space, choose one element to label as v if $\langle v|v\rangle > 0$ and label the remaining element u. This includes all cases except for both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Using the Gram-Schmidt construction, the element

$$z = u - \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v$$

is orthogonal to v.

$$\langle z|v\rangle = 0.$$

From the properties (191) and the construction of z,

$$\begin{aligned} \langle u|u\rangle &= \langle z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v | z + \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v\rangle \\ &= \langle z|z\rangle + \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle} \\ &\geq 0. \end{aligned}$$

But $\langle z|z\rangle \geq 0$ and it follows that

$$\langle u|u\rangle \geq \frac{|\langle u|v\rangle|^2}{\langle v|v\rangle}.$$

This demonstrates the inequality unless both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. If both are zero, the properties (191) provide that

$$\begin{aligned}\langle u+v|u+v\rangle &= \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle u|u\rangle \\ &= 2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \\ &\geq 0.\end{aligned}$$

Similarly, $\langle u-v|u-v\rangle \geq 0$ provides that

$$2\mathbb{R}\mathrm{e}(\langle u|v\rangle) \leq 0.$$

As a consequence, $\mathbb{R}\mathrm{e}(\langle u|v\rangle) = 0$. Similarly, $\langle u+iv|u+iv\rangle \geq 0$ and $\langle u-iv|u-iv\rangle \geq 0$ provide that $\mathbb{I}\mathrm{m}(\langle u|v\rangle) = 0$. Then,

$$\langle u|v\rangle = 0$$

if both $\langle v|v\rangle = 0$ and $\langle u|u\rangle = 0$. Summarizing, the Cauchy-Schwarz-Bunyakovsky inequality applies to a linear vector space with elements u, v and product $\langle u|v\rangle$ with properties (191).

If $\langle u|v\rangle$ is a scalar product and neither u nor v are the zero element, then

$$|\langle u|v\rangle|^2 = \langle u|u\rangle\langle v|v\rangle$$

if and only if

$$u = \alpha v$$

for a nonzero $\alpha \in \mathbb{C}$. For nonzero u and v and a scalar product, the equality applies if and only if $z = 0$ for the z constructed above and then

$$u = \frac{\langle u|v\rangle}{\langle v|v\rangle}\, v.$$

6.2.4 Entanglement

Quantum mechanics includes descriptions of *entangled* states. Descriptions of states are elements of Hilbert spaces and elements are expressible as linear combinations of other elements. The linear expansion of states is a property of quantum physics distinct from classical physics.

A simple example of entanglement has four states: $|$up, up$\rangle$, $|$up, down$\rangle$, $|$down, up$\rangle$, and $|$down, down$\rangle$ that span the spin states of two spin one-half bodies. The states describe the

four possibilities for the measurement of spin polarizations along a particular axis. Any linear combination of these four states describes a state. For example,

$$\frac{1}{\sqrt{2}}|\text{up, down}\rangle + \frac{1}{\sqrt{2}}|\text{down, up}\rangle$$

is one possible state. For this example state, the polarizations are entangled: up for the first body occurs only with down for the second body, and down for the first body occurs only with up for the second body. The state of the first body is determined by the state for the second and total spin adds to zero in any observation. However, the states are not determined, the first body can be detected as either spin up or spin down with equal likelihoods in this example. But, knowledge of one of the spins implies knowledge of the other due to an entanglement of states. Entanglement of the spins is established when the particles are causally related; the entanglement persists as the particles become acausally separated. An entanglement of bodies need not be perfect. For example,

$$a\,|\text{up, up, down}\rangle + b\,|\text{up, down, up}\rangle + c\,|\text{down, up, up}\rangle$$

is an example with a particular angular momentum but no perfectly entangled pair. In appendix 6.2.7, the states that result from an observation are described by linear combinations of states of an observer entangled with the possible results of observation, $|\psi\rangle = \sum_k c_k\,|o_k, s_k\rangle$. The o_k are orthogonal descriptions of the observer's state, and the s_k describe the entangled, observed state. In the example of Schrödinger's cat thought experiment [51], o_1 would be "observed a live cat," o_2 would be "observed a dead cat," s_1 would be "a live cat," s_2 would be "a dead cat" and these descriptions are entangled $c_1|o_1, s_1\rangle + c_2|o_2, s_2\rangle$. An appeal to experience indicates that $|o_1, s_2\rangle$ and $|o_2, s_1\rangle$ do not persist in the evolution of states.

6.2.5 Hilbert space operators

Hilbert space operators are linear maps of elements of a Hilbert space $\mathbf{H}$ to elements within the Hilbert space [3, 40]. A is a *Hilbert space operator* if

$$A : |\underline{f}\rangle \in \mathbf{H} \mapsto |A\underline{f}\rangle \in \mathbf{H}$$

for a subset of elements $|\underline{f}\rangle \in \mathcal{D}_A \subseteq \mathbf{H}$ denoted the domain of A. For a *bounded* operator,

$$\|A\psi\| \leq \|A\|\,\|\psi\|$$

with

$$\|A\| := \sup(\|A\psi\| : \|\psi\| \leq 1) < \infty.$$

A bounded operator is continuous, and the domain can be extended to the entire Hilbert space. Unitary operators are bounded and preserve the scalar product, that is, for unitary U,

$\langle Ug|U\psi\rangle = \langle g|\psi\rangle$. Illustrated in section 5, an *unbounded* operator is not continuous, its domain is not a (closed) subspace, and the domain necessarily does not include the entire Hilbert space although the domain may be dense in the Hilbert space. The adjoint operator A^* of a Hilbert space operator A is defined by the property that $\langle A^*h|\psi\rangle := \langle h|A\psi\rangle$ for $|\psi\rangle \in \mathcal{D}_A$ and $|h\rangle \in \mathcal{D}_{A^*}$. The set of $h \in \mathbf{H}$ such that there is a $g \in \mathbf{H}$ with $\langle h|A\psi\rangle = \langle g|\psi\rangle$ is the domain $\mathcal{D}_{A^*}$ of A^*. An operator A with domain $\mathcal{D}_A$ is *Hermitian* if $\langle \underline{u}|A\underline{v}\rangle = \langle A\underline{u}|\underline{v}\rangle$ for every $\underline{u}, \underline{v} \in \mathcal{D}_A$ [3], a Hermitian operator is *symmetric* if $\mathcal{D}_A$ is dense in the Hilbert space, and a symmetric operator is *self-adjoint* if $\mathcal{D}_A = \mathcal{D}_{A^*}$ and $A\underline{u} = A^*\underline{u}$ for every $\underline{u} \in \mathcal{D}_A$. These designations correspond to Hermitian, maximal Hermitian, and hypermaximal respectively in von Neumann's designation [60].

6.2.6 Operators in quantum mechanics

Every closed subspace of states corresponds with a projection operator [40]. Fundamental results for Hilber psace operators include the spectral theory for rigged Hilbert space operators [21, 24, 40]; and Stone's theorem [24] for unitarily realized groups of Hilbert space transformations. Spectral theory displays the Hermitian operators that represent observables as limits of linear combinations of projections weighted by representative values of the observable associated with each subspace of states. Projection operators E are bounded, self-adjoint and idempotent, $E = E^* = E^2$, $\|E\psi\| \leq \|\psi\|$. For projections, the range $\mathcal{R}_E = E\mathbf{H} \subset \mathbf{H}$, a proper subset unless $E = \mathbb{I}$. From a projection E, any state $|\psi\rangle \in \mathbf{H}$ may be decomposed as an element $|E\psi\rangle$ within the subspace $E\mathbf{H}$ and an element $|(\mathbb{I} - E)\psi\rangle$ in the orthogonal complement of the subspace, Riesz's theorem.

The states of nature are described by nonnegative ($\langle\psi|\rho\psi\rangle \geq 0$), self-adjoint ($\rho = \rho^*$, $\mathcal{D}_\rho = \mathcal{D}_{\rho^*}$ dense in $\mathbf{H}$), trace-class, normalized operators ρ (Trace$[\rho] = 1$).

$$\text{Trace}[\rho] := \sum_k \langle e_k|\rho e_k\rangle \tag{192}$$

if the $|e_k\rangle$ are an orthonormal basis for the separable Hilbert space $\mathbf{H}$. This basis is not unique and every unitary operator maps a basis to a basis, $|Ue_k\rangle = |e'_k\rangle$. $U^* = U^{-1}$ for a *unitary* operator and unitary operators are bounded. These operators ρ are designated *state density operators* [60] and generalize the vector states generally discussed above. *Vector states* have state density operators $\rho = E$ for E the projection onto a single element $|\psi\rangle$ ($\mathcal{R}_E = \{|a\psi\rangle : a \in \mathbb{C}\}$). Born's rule includes: if a state is described by the state density operator ρ, then the likelihood of observing a state in the subspace $E\mathbf{H}$ corresponding to a projection E given an initial state described by ρ is Trace$[E\rho] \leq 1$.

Operators associated with observables have mean values

$$E[A] := \text{Trace}[A\rho] \in \mathbb{R}$$

for Hermitian operators A and state density operators ρ. Measurements are real numbers and the expectation values of Hermitian operators are real. If A is Hermitian, $A = A^*$ on the domain of A, then for every $\underline{f} \in \mathcal{D}_A$,

$$\begin{aligned}\langle A^*\underline{f}|\underline{f}\rangle &= \langle A\underline{f}|\underline{f}\rangle \\ &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle \underline{f}|A\underline{f}\rangle\end{aligned}$$

from the properties of scalar products and definition of the adjoint operator. Then $A = A^*$ implies that the expectation values $\langle \underline{f}|A\underline{f}\rangle$ are real. And, if the expectation value is real and $\underline{f} \in \mathcal{D}_A$, then

$$\begin{aligned}\langle \underline{f}|A\underline{f}\rangle &= \overline{\langle \underline{f}|A\underline{f}\rangle} \\ &= \langle A\underline{f}|\underline{f}\rangle\end{aligned}$$

and $A = A^*$ for $\underline{f} \in \mathcal{D}_A$. As a consequence, observables are limited to Hermitian operators.

Significant examples of observables in nonrelativistic quantum mechanics are the Hermitian location operators X_ν. In $\mathcal{L}^2$,

$$\langle \psi|\psi\rangle = \int d\mathbf{x}\ |\psi(x)|^2$$

is finite at each time t with $x = ct, \mathbf{x}$. The summation is over three dimensional space, $\mathbb{R}^3$. Three location operators X_ν, one for each of the three spatial dimensions, are given by

$$X_\nu\psi(x) := x_\nu\psi(x).$$

There is no bound on the values assumed by location in a Euclidean space and consequently the location operator is an unbounded operator. $x_\nu\psi(x)$ is not necessarily square-integrable for square-integrable $\psi(x)$ and as a consequence, $\mathcal{D}_{X_\nu} \subset \mathbf{H}$, a proper subset. Functions $\psi(x) \in \mathcal{L}^2$ such that $x_\nu\psi(x) \in \mathcal{L}^2$ are dense in $\mathcal{L}^2$ but elements of slow decline are excluded from $\mathcal{D}_{X_\nu}$ (e.g., $\psi(x) = (1 + \mathbf{x}^2)^{-1+\delta} \in \mathcal{L}^2$ for $1 \gg \delta > 0$ in three dimensions but $x_\nu\psi(x) \notin \mathcal{L}^2$).

6.2.7 Observation in Hilbert spaces

The Everett-Wheeler-Graham relative state interpretation of the formalism of quantum mechanics [11] escapes the measurment paradoxes and ad hoc assertions of earlier understandings. The development describes observation from the premise that the "mathematical formalism of quantum mechanics is sufficient as it stands" [11]. This interpretation becomes necessary to realize relativistic quantum mechanics. Nonrelativistic concepts such as "collapse of the wave function to an eigenfunction" are inconsistent with relativity. It would take a period of infinite

duration for the support of a typical state describing function such as an energy eigenfunction of the hydrogen atom to collapse to a single point. The nonzero support of the energy eigenfunction is infinite: the speed of light is finite. Causality requires that the time evolution of state describing functions is expressed with operators that satisfy the Poincaré invariance of likelihoods. Time evolution includes measurements. With the observer as well as the observed included in the quantum mechanical description, time evolution is continuous, unitary time translation and there is no need to distinguish what interactions constitute measurement. The relative state description is consistent with our experience as well as consistent with the quantum mechanical description of nature.

The process of observation is fundamental in quantum mechanics: observers are not omniscient and the interactions that constitute measurements affect the relevant description of the state. This process explains how the perception of nature can differ from the description.In classical physics, the state can be observed without disruption: an observer is external to the systems under consideration and is essentially omniscient. However, in quantum mechanics, observation is an interaction within the quantum description: an observation entangles descriptions of the possible states of an observer with the observed. Observers are also described by quantum mechanics. There is no "macroscopic" or "classical" domain governed by distinct physical principles. A quantum description of state that describes a composite of system under observation and observer evolves into a linear combination that describes distinct alternative possibilities for observation. For example, an initially localized state spreads over space with time and a distinct state of the observer becomes entangled with each possible outcome of a subsequent localizing interaction. It is inconsistent with our experience that there is only one observer state: observers perceive results from among possibilities. To be included in a quantum description and avoid the difficulties described in the Schrödinger cat and EPR measurement paradoxes [51, 16], a distinct state of the observer must be entangled with each of the possibilities for observation. Distinct perceptions are described by distinct states. Within linear expansions of state descriptions, "we" are described by one of the possible states. This relative state or "many worlds" description [11] derives naturally from the quantum mechanical description of state as elements of Hilbert spaces. Although inherent to quantum mechanics, this natural interpretation is very different from classical concepts and it took several decades after the initial formulations of quantum mechanics for the relative state interpretation to emerge. Hugh Everett III, John Archibald Wheeler, Neill Graham and Bryce DeWitt developed the relative state interpretation of quantum mechanics [11]. Earlier interpretations maintained classical descriptions for observers. Even the designation as "many worlds" betrays a classical predisposition: many classical worlds but a unified, consistent quantum description. Consideration of the state as a superposition over many classical worlds, each with a state described by classical dynamical variables is an example of a persistent classical predisposition. The relative state interpretation uses the natural quantum mechanical description of states by functions with entanglement leading to a decomposition into relative states labeled by the perceived results of observations.

The quantum mechanical description of observations also explains our comfort with the classical view: often, within the precision of our perceptions, the resulting quantum mechanical description appears to replicate a classical description. However, a richer range of descriptions is included among the descriptions of state in quantum mechanics. A localized interaction results in observation of a localized state; two slits in a screen followed by localizing detectors results in observation of wave-like interference when the initial states are widely supported and propagate toward the screen. In common instances, the perception of a quantum description is indistinguishable from a classical description although in general, in instances such as the uniformly illuminated double slit, a quantum description of state deviates substantially from a classical description. Quantum mechanics supersedes classical descriptions.

The states of an observer are labeled by their perceptions. The relevant final state of the observed is determined by the entangled observer state, the result of an interaction characterized as an observation. The likelihoods of the various possibilities are determined by scalar products, the relevant initial state description, the description of perceived state, and chance. Chance determines our particular perception from among the possibilities. Perceptions such as the track of a planet may be the overwhelming likelihood within our measurment uncertainties, or the likelihoods may deviate substantially from any classical descriptions such as for Young's double slit or the energy levels of an atom. Observation results in a random selection of our perceived state from within a linear expansion of the temporal propagation of the initial state. If $|\underline{\psi}\rangle$ describes an initial state including the description of our history of perceptions, then an interaction of the observer and a system will result in an expansion of the evolved state. The states in the expansion are states labeled by the possibilities for updated histories of the observer's perceptions,

$$|U(t)\underline{\psi}\rangle = \sum_{\theta} |E_\theta U(t)\underline{\psi}\rangle = \sum_{\theta}\sum_{k} \langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle \; |\underline{g}_{k,\theta}\rangle. \tag{193}$$

The initial state $|\underline{\psi}\rangle$ jointly describes the observer and observed, and the final state expands in entangled descriptions for each of the possible perceived results θ of the observation. Entanglement was introduced in appendix 6.2.1 and entanglement develops in the time evolution of a state that initially has independent descriptions for an observer and an observed. Consistently with the separability of the Hilbert space, the notation in (193) is that this summation is denumerably indexed, our observations have finite resolution. And, the notation applies when the orthogonal subspaces associated with projections E_θ are spanned by a basis of elements $|\underline{g}_{k,\theta}\rangle$, $E_\theta = \sum_k |\underline{g}_{k,\theta}\rangle\langle \underline{g}_{k,\theta}|$. The $|\underline{g}_{k,\theta}\rangle$ jointly describe states of the observer and observed. The orthogonal projections E_θ are onto subspaces of the Hilbert space with descriptions of observers with a history of perceptions θ. Well designed measurements entangle descriptions that approximate eigenstates for the measured quantity[o] with the observer's states. Observers often

[o]Discussed in section 2, these eigenstates may be distinct from eigenstates of the quantizations of the classical dynamical variable.

interpret the body's state as an eigenstate of the quantization of the corresponding classical variable. That is, for example, in a well designed measurement, the observer interprets the perception of a state dominantly supported within a small neighborhood of $\mathbf{x}_o$ as a point-like body located at $\mathbf{x}_o$.

We never observe that recorded observations change, that further observations or communications change the record of past measurements or that the current state is inconsistent with past perceptions. This experience indicates that states labeled by distinct perceptions are orthogonal: the Hilbert space decomposes into orthogonal subspaces labeled by the distinct histories of observations. This decomposition provides no likelihood of transitions from our history of perceptions to a distinct history. The observation may be in error or imperfect, that is, the actual state entangled with an observer may not be the perceived state, but distinct histories of perceptions lie in orthogonal subspaces. Perception do not spontaneously change. Then,

$$E_\theta E_\vartheta = 0 \qquad \text{if} \qquad \theta \neq \vartheta,$$

and $\sum_\theta \; E_\theta = \mathbb{I}$ at least within an orthogonal subspace that includes the initial state $|\underline{\psi}\rangle$ and the states $|\underline{g}_{k,\theta}\rangle$ of interest. Whether this summation includes all possible states in the Hilbert space is not of issue. Our only concern is a particular subspace $|E_\theta U(t)\underline{\psi}\rangle$ that includes our particular history of perceptions, *us*.

The likelihood of adding the observation described by $\underline{g}_{k,\theta}$ to the history of the observer is given by Born's rule,

$$\text{Likelihood=Trace}(E_\theta \rho) = \sum_k |\langle \underline{g}_{k,\theta}|U(t)\underline{\psi}\rangle|^2$$

in the case of a normalized vector state $\rho := |\underline{\psi}\rangle\langle\underline{\psi}|$ and with E_θ the projection onto the subspace with orthonormalized basis $|\underline{g}_{k,\theta}\rangle$.

The Everett-Wheeler-Graham (EWG) interpretation includes a virtual collapse of the full description of nature to a *relative state* entangled with an observer's perception. Demonstrated in [11] and illustrated below, conditional predictions based solely on this relative state agree with predictions using the full description of state. This agrees with our experience; we need only know what we have observed and other possibilities are of no consequence to our ability to predict future outcomes. Alternative "worlds" have no reality for us. The alternatives do not affect our predictions although our experience is one from among many possibilities. This development resolves one of the mysteries of quantum mechanics: why do we never perceive a state in a superposition over contradictory states like a superposition of 'live cat' and 'dead cat' in Schrödinger's cat thought experiment [51]? Our observations entangle us with one or the other state. For the Schrödinger cat thought experiment, the descriptions of final states would be a linear combination of the $\underline{g}_{k,\theta}$ that describes observation of a live cat entangled with a live cat and an undecayed isotope, and an orthogonal state $\underline{g}_{j,\vartheta}$ that describes observation of a dead cat entangled with a dead cat and a decayed isotope. The formalism of quantum

mechanics is adequate to describe our experience. There is no need to augment quantum mechanics with state collapse as well as unitary evolution. Early interpretations of quantum mechanics included ad hoc assertions like "collapse of the wave packet" or "hidden variables" in attempts to preserve a classical understanding. Further development has demonstrated that our observations of nature are adequately described by quantum mechanics without such ad hoc augmentations.

Preservation of a "classical domain" are less disruptive to established understanding but incur the famous measurement paradoxes that illustrate the need for a quantum mechanical treatment of measurement. These classical measurement paradoxes include the Einstein-Podolsky-Rosen (EPR) [16], Schrödinger's cat [51], and Wigner's friend [65] paradoxes. The EPR paradox is discussed in appendix 6.6 and illustrates that quantized, conserved quantities can not be classically described. Schrödinger's cat paradox illustrates that the quantum description can not be relegated to a microscopic world and Wigner's friend illustrates the arbitrary nature of assertions necessary to descriptions of measurement as "collapse of the wave packet." The paradoxes emerge from the adoption of contradictory concepts: a classical description for observer with a quantum formalism for dynamics.

The EWG interpretation of quantum mechanics develops the concept of relative (entangled) state. A Hermitian operator A corresponding to an observable has a real expectation values. The conditional expectation is the expected value with states limited to a subspace of the Hilbert space. A Hermitian rigged Hilbert space operator A is represented[p] as

$$A = \sum_k a_k E_k$$

with orthogonal projections E_k from a resolution of unity and $a_k \in \mathbb{R}$ [21, 40]. Eigenvalues a_k are the observed values that follow for states in the subspaces $E_k\mathbf{H}$. The conditional expectation value of the observable A conditioned upon an observer's perception θ is

$$\begin{aligned} E^\theta[A] &:= \frac{E[E_\theta A E_\theta]}{E[E_\theta]} \\ &= \frac{\text{Trace}(E_\theta A E_\theta \rho)}{\text{Trace}(E_\theta \rho)} \end{aligned} \tag{194}$$

with ρ the nonnegative, unit trace, state density operator [60] discussed in appendix 6.2.6. The E_θ are the orthogonal projections (193) that project onto subspaces of states that include the observer perceptions labeled by history θ entangled with states of the observed. The E_θ project onto composites of observed plus observer states. Two example constructions of composite states are discussed below.

From the idempotence of projections and transpositional invariance of the trace (192) [27], Trace(AB)=Trace(BA),

$$\text{Trace}(E_\theta \rho) = \text{Trace}(E_\theta \rho E_\theta)$$

[p] Appropriate limits of these summations are included.

with $E_\theta \rho E_\theta$ the projection of the state density operator into the subspace labeled by θ. The *relative state density operator* [11], relative to the observer history labeled θ, is

$$\rho^\theta := \frac{E_\theta \rho E_\theta}{E[E_\theta]} \tag{195}$$

normalized to unit trace. The relative state density operators are orthogonal, $\rho^\theta \rho^\vartheta = 0$ if $\theta \neq \vartheta$, and remain orthogonal with time evolution implemented by unitary time translation, $U(t)^* U(t) = 1$. In terms of this relative state density operator, the *conditional expectation* is

$$E^\theta[A] = \text{Trace}(A\rho^\theta).$$

It was argued on physical grounds that states labeled by different histories of perceptions were orthogonal since it is our experience that there is no likelihood of a change to history. As a consequence, there will be a resolution of unity into projection operators E_θ labeled by the perception histories θ.

$$\sum_\theta E_\theta = 1$$

with the θ labeling the possible histories of perceptions, plus one additional orthogonal subspace of all remaining state descriptions. Distinguish this latter projection as E_0. Decomposition of the Hilbert space into orthogonal complements follows from the orthogonality of states with distinct histories, and Riesz's theorem on orthogonal subspaces in a Hilbert space [48].

If the operator A corresponding to an observable quantity is in the commutant of the E_θ from (194), that is, if

$$[A, E_\theta] = 0$$

for each E_θ except possibly with an exception for E_0, then a mixture of the relative state density operators ρ^θ is equivalent to the state density operator ρ. The resolution of unity $\sum E_\theta = 1$, the idempotence of projections, commutation of A and the E_θ, the transposition invariance of the trace (192) and the definition of relative state results in the equivalence.

$$\begin{aligned}
E[A] &= \text{Trace}(A\rho) \\
&= \sum_\theta \text{Trace}(E_\theta A \rho) \\
&= \sum_\theta \text{Trace}(E_\theta^2 A \rho) \\
&= \sum_\theta \text{Trace}(E_\theta A E_\theta \rho) \\
&= \sum_\theta \text{Trace}(A E_\theta \rho E_\theta) \\
&= \sum_\theta \text{Trace}(A\rho^\theta) E[E_\theta] \\
&= \sum_\theta E^\theta[A] E[E_\theta].
\end{aligned}$$

Then, using the definition of relative state density operator (195), if an operator A is in the *commutant* of the E_θ, the mixture of relative states,

$$\rho^{eq} = \sum_\theta E[E_\theta]\,\rho^\theta$$

has the same observable properties as the complete state description ρ.

$$\text{Trace}(A\rho) = \text{Trace}(A\rho^{eq}). \tag{196}$$

Each of the relative state density operators ρ^θ provide the expectation values conditioned on a history of observer perceptions θ. The subspace $E_0\mathbf{H}$ is of no interest. This equivalence of a mixture of the mutually orthogonal relative state density operators with the complete description of the state ρ explains why the quantum mechanical description of state is not in contradiction with a classical concept for an observer. Each term in a mixture evolves independently in time: knowledge of the entire state description is not required to propagate the relative states forward in time. Conditional expectations are equivalent to distinguishing one particular history, our history, as a classical observer. All our future observations evolve from our current relative state description ρ^θ. We need not know or account for the "other branches" to predict the future that is relevant to us, the future conditioned on our history. This suggests the "collapse of the wave function" upon observation described in early developments of quantum mechanics [60], but there is no physical distinction between a virtual and actual collapse of the state description ρ to ρ^θ upon observation. An actual collapse would result in a classical description of the observer but the universality of likelihood conservation, that is, a unitary implementation of time translation, is preserved if the collapse is considered as virtual. The equivalence remains with forward time translation.

$$U(t)\rho\,U(t)^* = \sum_\theta E[E_\theta]\,U(t)\rho^\theta\,U(t)^*.$$

The first example with system observables A and projections E_θ that commute is a tensor product of two Hilbert spaces. The states of the observed are elements of a Hilbert space $\mathbf{H}_1$ and the states of the observer are elements of a Hilbert space $\mathbf{H}_2$. The Hilbert space of interest is the tensor product, $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2$. A second example of this commutation is developed in appendix 6.2.8 and is based on the strong cluster decomposition property of the VEV. The second example describes a body that is initially distantly spatially separated from the observer and the commutation is with arbitrarily great likelihood, but inexact since there are no state descriptions of bounded support within $\mathbf{H}_\mathcal{P}$. In both cases, the observables A commute with the projections E_θ.

A *tensor product* is the composition of Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} := \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From descriptions of the body $\psi, g \in \mathbf{H}_1$ with scalar product $\langle\psi|g\rangle_1$, and states of the observer described $u, v \in \mathbf{H}_2$ with scalar product $\langle u|v\rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ has states labeled $\psi \otimes u, g \otimes v$ with scalar product

$$\langle\psi \otimes u|g \otimes v\rangle := \langle\psi|g\rangle_1 \langle u|v\rangle_2.$$

If ${}^1e_k$ and ${}^2e_k$ are orthonormal bases for separable $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively, then an orthonormal basis for the composite Hilbert space has labels

$$e_{kj} = {}^1e_k \otimes {}^2e_j$$

and $\dim(\mathbf{H}) = \dim(\mathbf{H}_1) \times \dim(\mathbf{H}_2)$. Operators defined in $\mathbf{H}_1 \otimes \mathbf{H}_2$ include extensions of the operators from $\mathbf{H}_1$ and $\mathbf{H}_2$. Hilbert space operators in the tensor product include $C := A \otimes B$,

$$\langle\psi \otimes u|C\ g \otimes v\rangle := \langle\psi|Ag\rangle_1 \langle u|Bv\rangle_2.$$

Then any operators $A \otimes \mathbb{I}$ commute with any $\mathbb{I} \otimes B$. If $\mathbf{H}_1$ includes the descriptions of the observed and $\mathbf{H}_2$ includes the descriptions of the observer, then this example satisfies the assertions above in discussion of the physical equivalence of the mixture of relative states ρ^θ to ρ. For an observation to occur, the time translation $U(t)$ from (193) must couple the constituent Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$. Tensor products demonstrate that the assumed concept of observables for a body in the commutant of the projections onto observer states is realizable. $\mathcal{B}(\mathbf{H}_1) \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{B}(\mathbf{H}_2)$ demonstrate that there are commuting sets of operators of interest. $\mathcal{B}(\mathbf{H})$ is the set of all bounded operators for the Hilbert space $\mathbf{H}$. That the tensor product composition suffices to define a Hilbert space is discussed further in appendix 6.2.9. A simple example illustrating entanglement and the commuation of observables with projections onto the observer states is in appendix 6.15.

The equivalence of ρ^{eq} and ρ provides that there is no need for a collapse of state description upon measurement. The relative state interpretation is the natural understanding of quantum mechanics, bizarre images summoned by "many worlds" notwithstanding.

6.2.8 Localized observables, separation and independence

The intuitive notion that separation implies independence provides a second example of Hermitian operators A associated with the observed that (nearly) commute with projections E_θ onto states that include a particular history of observer perceptions. Cluster decomposition (27) provides that scalar products factor when the support of the functions that describe the observed and observer are greatly space-like separated. This is suggestive of a tensor product decomposition of Hilbert spaces.

This intuitive, satisfying property that separation imples independence is enabled by consideration of localized observations. Observables essentially limited to bounded, isolated spatial volumes associate observables with bodies. And, if the observed is greatly spatially isolated

from the observer, then these observables (arbitrarily nearly) commute with projections onto states that describe the observer. The approximate independence is that

$$0 \approx \|[E_\theta, A]\| \ll \|A\|$$

with A a Hermitian operator representing the localized observable and for projections $\|E_\theta\| = 1$. This commutation is the property used to develop the equivalence of the mixture of relative states with a general state descriptions in the EWG development of observation discussed in appendix 6.2.7. The commutation is only approximate because the supports of state describing functions are not of bounded support: the functions within $\underline{\mathcal{P}}$ are anti-local. In these instances, the equivalence (196) of the mixture of relative states ρ^θ with ρ is to arbitrarily great likelihood.

In the constructions, the cluster decomposition condition (27) provides that the truncated functions defined by cluster expansion (71) are connected functions,

$$^C\mathcal{W}_{k,n-k} = {}^T\mathcal{W}_{k,n-k}.$$

The vanishing of connected functions $^C\mathcal{W}_{k,n-k}$ as the supports of arguments are greatly space-like separated, and satisfaction of cluster decomposition (27) provides that

$$\mathcal{W}_{n,m}((y)_{n+m}) = \mathcal{W}_{j,\ell}((x)_P)\mathcal{W}_{n-j,m-\ell}((x)_{P'})$$

as $\lambda \to \infty$ if $y_{i_k} = x_k$ for $i_k \in P$ and $y_{i_k} = x_k + (0, \lambda\mathbf{a})$ for $i_k \in P' = \{1, n+m\}/P$, the set complement of the $j+\ell$-element set of integers P from $\{1, n+m\}$, with the evident embellishment of the $(x)_n$ notation.

The demonstration is limited to a selected instance. The observed is an elementary particle described by a one-argument function $\psi(x)$. In the final, post measurement phase, the support of $\psi(x)$ is distantly spatially isolated from the support of the description of the observer. $\mathbf{x}_o$ is the time dependent location of the observed. A finite spherical volume centered on $\mathbf{x}_o$ is designated by $V_{\mathbf{x}_o}$. $V_{\mathbf{x}_o}$ includes all but a negligible amount of the support of $\psi(x)$. Assert that there is a choice of eigenfunctions $\{e_\ell(x)\}$ such that: the support of every $e_\ell(x)$ outside of $V_{\mathbf{x}_o}$ is uniformly negligible; and the Hermitian operator $\hat{A}$ associated with the observable is

$$|\hat{A}\underline{f}\rangle := (0, A_1 f(x_1), \ldots, \sum_{j=1}^{n} A_j f_n((x)_n), \ldots) \tag{197}$$

with

$$A_j f_n((x)_n) := \sum_\ell a_\ell e_\ell(x_j) \int dy_1 dy_2 \; W_2(y_1, y_2)\overline{e_\ell(y_1)} f_n(x_1, \ldots x_{j-1}, y_2, x_{j+1}, \ldots x_n).$$

W_2 is the two-point function, and $j \in [1, n]$. The a_ℓ are the eigenvalues of the one-argument subspace observable A associated with the eigenfunctions $e_\ell(x)$.

$$A = \sum_\ell a_\ell |e_\ell\rangle\langle e_\ell|$$

and the state describing function expands in a linear combination of eigenvectors

$$\psi(x) \approx \sum_{\ell} \langle \underline{e}_\ell | \underline{\psi} \rangle \; e_\ell(x). \tag{198}$$

The functions $e_\ell(x)$ are orthonormal,

$$\langle \underline{e}_\ell | \underline{e}_{\ell'} \rangle = \delta_{\ell,\ell'}. \tag{199}$$

The single argument operator A_j is A applied to the jth argument in the n-argument subspace. A is asserted to be bounded and essentially localized within the volume $V_{\mathbf{x}_o}$. A is designated *essentially localized* within $V_{\mathbf{x}_o}$ if cluster decomposition (27) implies that

$$\langle \underline{f} | \hat{A} \underline{f} \rangle \to 0 \tag{200}$$

if the dominant support of a localized state $|\underline{f}\rangle$ is arbitrarily greatly space-like separated from $V_{\mathbf{x}_o}$. These properties restrict the class of observables. For example, the linear harmonic oscillator energy eigenfunctions are localized but not uniformly negligible outside of a bounded volume: the greater the energy, the broader the support. The volume $V_{\mathbf{x}_o}$ is selected so that the contribution of any $\psi(x)$ or $e_\ell(x)$ of interest beyond $V_{\mathbf{x}_o}$ is arbitrarily negligible. Examples of localized, one argument observables include location with the $e_\ell(x) \approx \delta(\mathbf{x} - \mathbf{x}_\ell)$ and $x_\ell \in V_{\mathbf{x}_o}$.

The second set of operators of interest are projections E_θ onto states that include particular descriptions of the observer. The initial description of the observer is selected to be (nearly) independent of the description of the observed: initially the observer and observed are unentangled with distantly space-like separated supports. E_θ projects a joint, final, post-observation description of observer and observed (193) onto a description with the observer described by a particular new perception added to the history θ. The final state of the evolved observer state is labeled $\underline{h}_0$ and the final observer states with the evolved perceptions are labeled $\underline{h}_\theta$. These descriptions of state do not include description of the observed. The supports of both the $\underline{h}_0$ and $\underline{h}_\theta$ within $V_{\mathbf{x}_o}$ are arbitrarily negligible. Initial and final are long before and long after the interaction that constitutes an observation. The projections onto the final observer states labeled $\underline{h}_\theta$ are

$$E_\theta := \text{orthogonal projection onto the union of the ranges of } E_{\underline{g}_{\ell\theta}} \tag{201}$$

defined from projections

$$|E_{\underline{g}_{\ell\theta}} \; \underline{f}\rangle := \frac{\langle \underline{g}_{\ell\theta} | \underline{f} \rangle}{\langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta} \rangle} \, |\underline{g}_{\ell\theta}\rangle$$

onto the product states defined

$$|\underline{g}_{\ell\theta}\rangle := |\underline{e}_\ell \times \underline{h}_\theta\rangle \tag{202}$$

with $\underline{e}_\ell := (0, e_\ell(x_1), 0, \ldots)$ the eigenfunctions of the observable.

Estimates for the commutator of the observed's essentially localized observable A with the projections E_θ onto particular observer states follow from the initial independence, the final distant spatial separation, and cluster decomposition (27). Demonstrated immediately below, the definitions for the observables (197), (199), (198), and (200), projections (201), and joint observer-observed states (202) in the rigged Hilbert spaces of relativistic quantum physics provide that

$$\begin{aligned} |E_\theta \, \underline{\psi} \times \underline{h_0}\rangle &\approx \langle \underline{h}_\theta | \underline{h_0} \rangle \; |\underline{\psi} \times \underline{h}_\theta\rangle \\ |\hat{A} \, \underline{\psi} \times \underline{h_0}\rangle &\approx |(A\underline{\psi}) \times \underline{h_0}\rangle. \end{aligned} \tag{203}$$

The projection onto particular observer states does not affect the description of the observed. The likelihood of a particular final state of the observed is determined by the likelihood that the evolved $\underline{h}_0$ has a nonzero projection onto $\underline{h}_\theta$. The likelihoods of observed values for the localized observable $\hat{A}$ is determined only by the description of the observed.

The demonstration of (203) follows from the scalar product (21).

$$\begin{aligned} \langle \underline{g}_{\ell\theta} | \underline{\psi} \times \underline{h_0}\rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{\psi} \times \underline{h_0}\rangle \\ &\approx \langle \underline{e}_\ell | \underline{\psi}\rangle \; \langle \underline{h}_\theta | \underline{h_0}\rangle \end{aligned}$$

as a consequence of the distant space-like final separation of supports and cluster decomposition (27) of the VEV into connected functions. The separation of the volume $V_{\mathbf{x}_o}$ from the final support of the observer also provides that

$$\begin{aligned} \langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta}\rangle &= \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{e}_\ell \times \underline{h}_\theta\rangle \\ &\approx \langle \underline{e}_\ell | \underline{e}_\ell\rangle \langle \underline{h}_\theta | \underline{h}_\theta\rangle \\ &= 1 \end{aligned}$$

with the normalization $\|\underline{h}_\theta\| = 1$. Similarly, $\langle \underline{g}_{\ell\theta} | \underline{g}_{\mu\theta}\rangle \approx 0$ if $\ell \neq \mu$. This approximate orthogonality then provides that

$$E_\theta \approx \sum_\ell E_{\underline{g}_{\ell\theta}} \tag{204}$$

from (201). As a consequence,

$$\begin{aligned}
|E_\theta \underline{\psi} \times \underline{h_0}\rangle &\approx \sum_\ell |E_{\underline{g}_{\ell\theta}} \underline{\psi} \times \underline{h_0}\rangle \\
&= \sum_\ell \frac{\langle \underline{g}_{\ell\theta} | \underline{\psi} \times \underline{h_0}\rangle}{\langle \underline{g}_{\ell\theta} | \underline{g}_{\ell\theta}\rangle} \, |\underline{g}_{\ell\theta}\rangle \\
&= \sum_\ell \langle \underline{e}_\ell \times \underline{h}_\theta | \underline{\psi} \times \underline{h_0}\rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\
&\approx \sum_\ell \langle \underline{e}_\ell | \underline{\psi}\rangle \, \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\underline{e}_\ell \times \underline{h}_\theta\rangle \\
&= \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\underline{\psi} \times \underline{h}_\theta\rangle
\end{aligned}$$

from substitution of the expansion (198) of the observed state. E_θ approximates a projection operator with the desired property of projecting onto the observer state of interest independently of the descriptions of the observed.

The form (197) of the local observable $\hat{A}$ provides

$$\begin{aligned}
\hat{A}\,\underline{\psi} \times \underline{h_0} &= (0, A\psi(x_1) f_{o,0}, \ldots \sum_{j=1}^{n} A_j \psi(x_1) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\
&\approx (0, A\psi(x_1) f_{o,0}, \ldots (A\psi(x_1)) f_{o,n-1}(x_2, \ldots x_n), \ldots) \\
&= (A\,\underline{\psi}) \times \underline{h_0}
\end{aligned}$$

as a consequence of the distant spatial separations of the support of the $e_\ell(x)$ and $\underline{h_0}$, and cluster decomposition. This completes the demonstration of (203).

From (197), the commutators of the $\hat{A}$ and E_θ follow from

$$\begin{aligned}
|\hat{A} E_\theta \, \underline{\psi} \times \underline{h_0}\rangle &\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |\hat{A} \, \underline{\psi} \times \underline{h}_\theta\rangle \\
&\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |(A\underline{\psi}) \times \underline{h}_\theta\rangle
\end{aligned}$$

and

$$\begin{aligned}
|E_\theta \hat{A} \, \underline{\psi} \times \underline{h_0}\rangle &\approx |E_\theta \, (A\underline{\psi}) \times \underline{h_0}\rangle \\
&\approx \langle \underline{h}_\theta | \underline{h_0}\rangle \, |(A\underline{\psi}) \times \underline{h}_\theta\rangle.
\end{aligned}$$

Then,

$$[E_\theta, \hat{A}] \approx 0$$

for states that describe an observed and observer that are initially unentangled and distantly space-like separated both initially and finally.

This development is not general but illustrates that separation and a lack of entanglement provide independence. This demonstration is an alternative to the tensor product example. This result is enabled within relativistic quantum physics by consideration of observations that are essentially localized, the likely association of locations with the observed, and satisfaction of cluster decomposition (27) in A.6.

6.2.9 Tensor products of linear vector spaces

Discussed in [60] and appendix 6.2.7, a *tensor product* is the composition of linear vector spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ into a composite

$$\mathbf{H} := \mathbf{H}_1 \otimes \mathbf{H}_2.$$

From elements described $g_1, g_2 \in \mathbf{H}_1$ with degenerate scalar product $\langle g_1|g_2\rangle_1$, and elements described $f_1, f_2 \in \mathbf{H}_2$ with degenerate scalar product $\langle f_1|f_2\rangle_2$, the tensor product composite Hilbert space $\mathbf{H}$ includes states labeled $g_1 \otimes f_1, g_2 \otimes f_2$ with degenerate scalar product

$$\langle g_1 \otimes f_1|g_2 \otimes f_2\rangle := \langle g_1|g_2\rangle_1 \langle f_1|f_2\rangle_2. \tag{205}$$

In this appendix, the sufficiency of this assignment to specify a degenerate scalar product on the complete tensor product space is addressed. (205) is evidently a degenerate scalar product of product elements $g_1 \otimes f_1, g_2 \otimes f_2$, but does this degenerate scalar product extend to all elements of $\mathbf{H}$?

First it is established that the Cauchy-Schwarz-Bunyakovsky inequality applies for all product states, and then this result is used to demonstrate that the degenerate scalar product applies to linear combinations of product states. Then, if $\mathbf{H}$ is defined as the completion of the linear span of product functions, then the degenerate scalar product applies to $\mathbf{H}$.

The Cauchy-Schwarz-Bunyakovsky inequality applies for all product states.

$$\begin{aligned} |\langle g_1 \otimes f_1|g_2 \otimes f_2\rangle|^2 &= |\langle g_1|g_2\rangle_1|^2 |\langle f_1|f_2\rangle_2|^2 \\ &\leq \langle g_1|g_1\rangle_1 \langle g_2|g_2\rangle_1 \langle f_1|f_1\rangle_2 \langle f_2|f_2\rangle_2 \\ &= \langle g_1 \otimes f_1|g_1 \otimes f_1\rangle \langle g_2 \otimes f_2|g_2 \otimes f_2\rangle \end{aligned}$$

from (205) and the Cauchy-Schwarz-Bunyakovsky inequality in the constituent linear vector

spaces. As a consequence, (205) extends to linear combinations of product functions.

$$\begin{aligned}|\langle u+v|u+v\rangle|^2 &= \langle u|u\rangle + \langle v|v\rangle + \langle u|v\rangle + \langle v|u\rangle \\ &= \langle u|u\rangle + \langle v|v\rangle + 2\Re e\langle u|v\rangle \\ &\geq \langle u|u\rangle + \langle v|v\rangle - 2\sqrt{\langle u|u\rangle\langle v|v\rangle} \\ &= \left(\sqrt{\langle u|u\rangle} - \sqrt{\langle v|v\rangle}\right)^2 \\ &\geq 0\end{aligned}$$

if the Cauchy-Schwarz-Bunyakovsky inequality applies to the elements u, v. It applies for product elements $g \otimes f$ and then nonnegativity applies to linear combinations of two product elements. Then, if the sum of two product elements has a nonnegative scalar product, then the sum of three also does by a similar argument. Hence, all linear combinations of product elements are nonnegative. The degenerate scalar product extends to an complete tensor product space if the space is defined as follows.

A denumerable basis of elements e_{nm} is defined from products of basis elements for the constituent Hilbert spaces

$$e_{nm} := {}^1e_n \otimes {}^2e_m$$

with ${}^1e_n$ a basis for the separable $\mathbf{H}_1$ and ${}^2e_m$ a basis for the separable $\mathbf{H}_2$. Then (205) provides that

$$\langle e_{nm}|e_{k\ell}\rangle = \delta_{n,k}\delta_{m,\ell}$$

with Kronecker deltas. The e_{nm} are a basis for $\mathbf{H}_1 \otimes \mathbf{H}_2$ defined as the completion $\mathbf{H}$ of all linear combinations of product states $g \otimes f$ in the norm from (205). Nonnegativity extends to all convergent limits.

$$\langle u|u\rangle = \langle \sum_{n,m} a_{nm}e_{nm}| \sum_{k,\ell} a_{k\ell}e_{k\ell}\rangle = \sum_{n,m}\sum_{k,\ell} \overline{a_{nm}}a_{k\ell}\delta_{n.k}\delta_{m,\ell} = \sum_{n,m} |a_{nm}|^2$$

that is manifestly nonnegative.

6.3 Translations, location operators and relativistically invariant localized states

Hermitian Hilbert space operators that correspond to location are discussed in this appendix.

Discussed below, multiplication of the Fourier transforms $\tilde{f}_n((p)_n)$ of functions by p_ν generates translations in location and multiplication of the functions $f_n((x)_n)$ by x_ν generates translations in momenta. Translation invariance of the scalar product and Stone's theorem leads to the conclusion that momenta are densely defined Hermitian operators in nonrelativistic physics. However, in relativistic physics, the scalar product is not invariant to translations

in momenta. In a nonrelativistic development, the $\mathcal{L}^2$ scalar product is invariant to translations in momenta but in a relativistic development, the universality of the speed of light dictates a Lorentz invariant scalar product. As a consequence, multiplication by x_ν does not generate a unitarily implemented symmetry in a relativistic development. If multiplication by x_ν were a densely defined Hermitian operator, then translations in momentum would be unitarily implemented and the scalar product would be momentum translation invariant. Multiplication of functions by x_ν can not correspond to the location operator in relativistic physics since it cannot be Hermitian, and as developed in appendix 6.2.6, only Hermitian operators are associated with observables. It is demonstrated in section 5 that the operator X_ν that quantizes x_ν is not Hermitian. Despite its commutation with the momentum operators, the "position" operators X_ν are not Hermitian for the relativistic scalar product (21).

From the properties (18) of the Fourier transform, translation of a function corresponds to multiplication of the Fourier transform by e^{-ipa}. Recall that $px = p_0ct - \mathbf{k}\cdot\mathbf{x}$, a Lorentz invariant. For both free fields and the constructions, the Hamiltonian is $H = \hbar c p_0$ [33]. From the scalar product (21), translation of the function to $x - a$ corresponds to translation of a field by $x + a$. Fourier transforms correspond

$$e^{-ipa}\tilde{\psi}(p) \leftrightarrow \psi(x-a)$$

and then for sufficiently small $\|a\|$, Taylor theorem polynomial approximation of the exponential function and $\psi(x)$ provide that

$$(1 - ipa)\tilde{\psi}(p) \leftrightarrow \psi(x) - a\cdot\frac{d\psi(x)}{dx}.$$

Then the operator P_ν that corresponds to multiplication of the Fourier transform by p_ν is

$$P_\nu = -i\hbar g_{\nu\nu}\frac{d}{dx_\nu}.$$

The Minkowski signature g is (19). With

$$U(a)\psi(x) := \psi(x-a),$$

the translation invariance of the scalar product provides that translations $U(a)$ are unitary and the generators,

$$U(a) = \exp(-iap),$$

are the energy-momentum operators $P_\nu = \hbar p_\nu$. From Stone's theorem, the energy-momentum operators are densely defined and Hermitian. P_ν is evidenly unbounded. Similarly, multiplication of functions by x_ν generates translations in energy-momenta. Fourier transforms correspond

$$e^{iqx}\psi(x) \leftrightarrow \tilde{\psi}(p-q)$$

and then for sufficiently small $\|q\|$,

$$(1 + iqx)\psi(x) \leftrightarrow \tilde{\psi}(p) - q \cdot \frac{d\tilde{\psi}(p)}{dp}.$$

Then the operator X_ν that corresponds to multiplication of a function by x_ν is

$$X_\nu = ig_{\nu\nu}\frac{d}{dp_\nu}.$$

With

$$T(q)\tilde{\psi}(p) := \tilde{\psi}(p - q),$$

if the scalar product were energy-momentum translation invariant then the

$$T(q) = \exp(iqX)$$

would be unitary and its Hermitian generators would be the location operators X_ν. The correspondence of X_ν and P_ν with location and momentum respectively is established in the interpretation of the support of the arguments of functions as spacetime coordinates and of the Fourier transforms as energy-momenta in units of wavenumber. Considering only spatial coordinates, $\nu = 1, 2, 3$,

$$X_\nu = x_\nu, -i\frac{d}{dp_\nu} \qquad \text{and} \qquad P_\nu = i\hbar\frac{d}{dx_\nu}, \hbar p_\nu \tag{206}$$

that apply to functions or the Fourier transforms of functions, respectively. Signs are determined by the convention for the Fourier transform (16). The X_ν and P_ν given by (206) canonically commute. But the properties of X_ν and P_ν as Hilbert space operators depend on the Hilbert space realizations, that is, depend on the scalar product. In the Hilbert space $\mathcal{L}^2$ appropriate for nonrelativistic physics [52], the scalar product is both spacetime and energy-momentum translation invariant and X_ν and P_ν are both densely defined Hermitian operators. For a relativistic scalar product, only the P_ν are Hermitian since in relativistic physics, there is only translation invariance of the scalar product. Demonstrated in section 5, multiplication by x_ν is not realized as a Hermitian Hilbert space operator for the relativistically invariant scalar products of elementary particle states.

The identification of X_ν as the location operator and P_ν as the momentum operator in nonrelativistic physics results in the Born-Heisenberg-Jordan relation for their commutator.

$$[X_\nu, P_\nu] = -i\hbar.$$

This commutation implies the Heisenberg uncertainty relation as discussed in appendix 6.7 and a Baker–Campbell–Hausdorff identity implies that similarity transforms of the location

operators translate the eigenvalues.

$$\begin{aligned} e^{-ia_\nu P_\nu/\hbar} X_\nu e^{ia_\nu P_\nu/\hbar} &= X_\nu + \frac{ia_\nu}{\hbar}\,[X_\nu, P_\nu] \\ &= X_\nu + a_\nu. \end{aligned} \tag{207}$$

As a consequence, there is a simultaneous eigenfunction of the X_ν for every $\mathbf{a} \in \mathbb{R}^3$ given a function that is in the union of the null spaces of the X_ν, $\nu = 1, 2, 3$. From $X_\nu \psi_0(\mathbf{x}) = 0$,

$$\psi_{\mathbf{a}}(\mathbf{x}) := e^{ia_1 P_1/\hbar} e^{ia_2 P_2/\hbar} e^{ia_3 P_3/\hbar} \psi_0(\mathbf{x}) \qquad \text{satisfies} \qquad X_\nu \psi_{\mathbf{a}}(\mathbf{x}) = a_\nu \psi_{\mathbf{a}}(\mathbf{x}).$$

This result followed solely from the Born-Heisenberg-Jordan relation and there are many other differential operators $\widehat{X}_\nu$ that also canonically commute with the momentum operators P_ν. These differential operators $\widehat{X}_\nu$ also have eigenstates associated with every location in $\mathbb{R}^3$.

From Clairaut's theorem, the operations

$$\widehat{X}_\nu = -iu(p)\frac{d}{dp_\nu}u^{-1}(p) \tag{208}$$

mutually commute as long as $u(p)$ is twice continuously differentiable. The $\widehat{X}_\nu$ canonically commute with the energy-momentum operators,

$$[\widehat{X}_\nu, P_\mu] = -i\hbar\delta_{\nu,\mu}$$

with $\nu, \mu = 1, 2, 3$. Then, there is a class of differential operators $\widehat{X}_\nu$ parametrized by twice continuously differentiable functions $u(p)$ that satisfy the Born-Heisenberg-Jordan commutation relations $[\widehat{X}_\nu, P_\nu] = -i\hbar$ (213) and that mutually commute, $[P_\nu, P_\mu] = [\widehat{X}_\nu, \widehat{X}_\mu] = 0$. The operations potentially correspond with observables only if they are Hermitian in the scalar product of the Hilbert space realization of interest.

The location operators $\widehat{X}_\nu$ are determined by the Hilbert space realization of quantum mechanics. Three distinct sets of location operators are developed in this appendix: the elevation of x_ν that applies in a canonical quantization of nonrelativistic physics (206); an $\widehat{X}_\nu$ for the relativistic free field realized in Fock space; and an $\widehat{X}_\nu$ for the relativistic constructions based upon the basis function spaces $\underline{\mathcal{P}}$ from section 3.7. The eigenfunctions of these operators are labeled by locations $\mathbf{a} \in \mathbb{R}^3$, are orthogonal for distinct locations as a consequence of Hermiticity [3], and are dominantly supported near each location $\mathbf{a}$ although they are not of point support in the relativistic cases. In relativistic physics, the lack of negative energy support and Lorentz covariance of the states implies that Dirac delta functions are not generalized eigenfunctions of a Hermitian operator within the Hilbert space.

The evident correspondence of particles with state descriptions provides that location operators apply for descriptions of free fields [31, 61]. A free field decomposes into canonically commuting particle creation and annihilation operators; every state can be represented as a

linear combination of particle creation operators applied to the vacuum state. When there is interaction, as discussed in section 3.1, these location operators apply to states in the one particle subspace and more generally provide a location for indeterminant bodies. If high order connected functions contribute significantly, then states are not necessarily interpretable as classical particles and the location of a particle as an observable is not evident. The extension of a single particle operator to multiple particle states (15) is a second quantization that follows (197) for the constructions when interaction is lacking. The discussion of section 3.1 illustrates that general state descriptions are not associated with determined numbers nor types of particles. A particle interpretation is a classically inspired perception of the quantum description of state.

In (208), $u(p) = 1$ is the Hermitian location operator X_ν in the Hilbert space $\mathcal{L}^2$ applicable in nonrelativistic physics. The eigenfunctions of the X_ν are Dirac delta functions and ideally associated with classical locations. More generally, in nonrelativistic limits, $\|\hbar\mathbf{p}\| \ll mc$, $\widehat{X}_\nu \approx X_\nu$ if $u(p) \approx u(mc/\hbar, 0, 0, 0)$. However, as discussed in section 2, eigenfunctions for a Hermitian operator are orthogonal in the Hilbert space scalar product. The $u(p)$ in (208) suffice to select Hermitian operators $\widehat{X}_\nu$ in realizations of relativistic physics [43].

The Hermitian operator associated with location is in the form (212) derived from the association of classical bodies with states that are dominantly supported within small isolated volumes. From the discussion of Heisenberg's uncertainty principle in appendix 6.7, product states with factors

$$\tilde{f}(p) := \frac{(p_0 + \omega)\tilde{\varphi}(\mathbf{p})}{\sqrt{2\omega}}$$

have scalar products that coincide with the $\mathcal{L}^2$ scalar product of the $\tilde{\varphi}(\mathbf{p})$ in the one-particle subspace or more generally for states that are well described by classical particles. Using the translates (207), and a selection of Dirac delta sequences

$$\tilde{\psi}_0(\mathbf{p}) = \omega^{1/2} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \omega^{1/2}$$

with $L \to 0$ for $\varphi(\mathbf{x})$ result in the relativistically invariant localized functions of [43]. For the relativistic free field construction based upon the basis function spaces $\underline{\mathcal{P}}$ [31], the Hermitian operator that corresponds to location has the additional constraint that the eigenfunctions must not be supported on negative energy mass shells.

$$\tilde{\psi}_0(\mathbf{p}) = \frac{(p_0 + \omega)}{\sqrt{2\omega}} e^{-L^2(\mathbf{p}-\mathbf{w})^2} \to \frac{(p_0 + \omega)}{\sqrt{2\omega}}.$$

That is, $\widehat{X}_\nu : \underline{\mathcal{P}} \mapsto \underline{\mathcal{P}}$ is required. States labeled by distinct locations $\mathbf{a} \neq \mathbf{a}'$ become orthogonal as $L \to 0$ and then there is a (generalized) eigenfunction associated with every location $\mathbf{a} \in \mathbb{R}^3$. In both cases, the functions $\tilde{\psi}_\mathbf{a}(\mathbf{p})$ are appropriate test functions that label elements of the constructed Hilbert space when L is finite and nonnegative.

If $u(p) = \omega^{1/2}$ or $u(p) = (p_0 + \omega)/\omega^{1/2}$ with ω from (13), then the resulting $\widehat{X}_\nu$ is Hermitian in the one particle subspace for the relativistic scalar product. From (37),

$$\begin{aligned}\langle f_1|\widehat{X}_\nu g_1\rangle &= \int \frac{dp}{2\omega}\,\delta(p_0-\omega)\overline{\tilde{f}_1}(p)\left(-i\omega^{1/2}\frac{d}{dp_\nu}\,\omega^{-1/2}\tilde{g}_1(p)\right)\\ &= -i\int \frac{d\mathbf{p}}{2}\,\omega^{-1/2}\overline{\tilde{f}_1}(\omega,\mathbf{p})\frac{d}{dp_\nu}\,\omega^{-1/2}\tilde{g}_1(\omega,\mathbf{p})\\ &= i\int \frac{d\mathbf{p}}{2}\,\left(\frac{d}{dp_\nu}\,\omega^{-1/2}\overline{\tilde{f}_1}(\omega,\mathbf{p})\right)\omega^{-1/2}\tilde{g}_1(\omega,\mathbf{p})\\ &= \langle \widehat{X}_\nu f_1|g_1\rangle\end{aligned} \tag{209}$$

from integration by parts.

An alternative method to associate $\widehat{X}_\nu$ with location is to set $\widehat{X}_\nu$ equal to a Hermitian operator derived from X_ν.

$$\widehat{X}_\nu := \frac{1}{2}(X_\nu + X_\nu^*)$$

with the adjoint operators X_ν^* defined for the free field scalar product (209). Then

$$\begin{aligned}\widehat{X}_\nu\tilde{f}_1(p) &= -\frac{i}{2}\left(\frac{d}{dp_\nu}+\omega\frac{d}{dp_\nu}\omega^{-1}\right)\,\tilde{f}_1(p)\\ &= -ir\left(\tilde{f}_1'(p)-\frac{\omega'}{2\omega}\,\tilde{f}_1(p)\right)\\ &= -i\omega^{1/2}\frac{d}{dp_\nu}\,\omega^{-1/2}\tilde{f}_1(p)\end{aligned}$$

from $\langle X_\nu^* f_1|g_1\rangle := \langle f_1|X_\nu g_1\rangle$, (209), integration by parts, and with the prime designating differentiation with respect to p_ν. However, this method is not general. In this case of location, the adjoint shares a dense domain with X_ν. The adjoint of the field $\Phi(\underline{f})^*$ from (25) is generally undefined for the constructions and $\widehat{\Phi}(\underline{f}) = \frac{1}{2}(\Phi(\underline{f}) + \Phi(\underline{f})^*)$ is then not Hermitian.

The relativistic location eigenfunctions, the relativistically invariant localized functions, are orthogonal. The two-point function for a free field (37) is expressed using the Fourier transforms,

$$W_2(\psi^* g) = \int \frac{d\mathbf{p}}{2\omega}\;\overline{\tilde{\psi}(\omega,\mathbf{p})}\;\tilde{g}(\omega,\mathbf{p})$$

and then the relativistically invariant localized functions centered on distinct locations $\mathbf{a}$ and $\mathbf{a}'$ are orthogonal.

$$\begin{aligned}W_2(\psi_{\mathbf{a}}^*\psi_{\mathbf{a}'}) &= \tfrac{1}{2}\int d\mathbf{p}\; e^{-i\mathbf{p}\cdot(\mathbf{a}'-\mathbf{a})}\\ &= \tfrac{1}{2}(2\pi)^3\,\delta(\mathbf{a}-\mathbf{a}').\end{aligned}$$

To characterize the spacetime support of $\tilde{\psi}_{\mathbf{a}}(\mathbf{p})$, [43] uses a transform with a Lorentz covariant measure supported solely on the mass shell,

$$\begin{aligned} f_{\mathbf{a}}(x) &:= 2\lambda_c^2 \int \frac{dp}{(2\pi)^{3/2}}\, \theta(E)\delta\left(p^2 - \lambda_c^{-2}\right)\, e^{ipx}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) \\ &= \lambda_c^2 \int \frac{d\mathbf{p}}{(2\pi)^{3/2}}\, \frac{1}{\omega}\, e^{i\omega ct - i\mathbf{p}\cdot\mathbf{x}}\tilde{\psi}_{\mathbf{a}}(\mathbf{p}). \end{aligned}$$

In the limit $L \to 0$ and for $t = 0$, $\tilde{\psi}_{\mathbf{a}}(\mathbf{p}) = \sqrt{\lambda_c \omega}\, e^{i\mathbf{p}\cdot\mathbf{a}}$ and

$$f_{\mathbf{a}}(x) = F(r/\lambda_c)$$

with

$$F(r/\lambda_c) = c_F\, (\lambda_c/r)^{5/4} H^{(1)}_{5/4}(ir/\lambda_c), \tag{210}$$

a Hankel function $H^{(1)}_{5/4}(z)$, dimensionless real constant c_F, and $r^2 := (\mathbf{x} - \mathbf{a})^2$ [1, 43]. The dominant support of $F(r/\lambda_c)$ is near $r = 0$ due to the divergence as $r^{-5/2}$ at the origin and a rapid decline bounded by $\exp(-r/\lambda_c)$ at large r. λ_c is the reduced Compton wavelength. The relativistically invariant localized functions $f_{\mathbf{a}}(x)$ do not equal zero within any spatial neighborhood.

By the characterizations developed in section 2, states such as relativistically invariant localized states would typically be perceived as localized states despite their lack of bounded support. The relativistically invariant localized states are essentially localized within a volume of radius proportional to the Compton wavelength of the body, 3.0×10^{-13} m for an electron.

6.4 Location, a prototype correspondence in relativistic quantum physics

Location contradicts a canonical quantization in relativistic physics. Location is a classical dynamical variable represented by a spatial argument of a state describing function in a canonical quantization. Elevation of location to quantum mechanical operator is not Hermitian in relativistic physics [43, 66]. Location demonstrates that a canonical quantization is not generally available. One significant distinction between relativistic and nonrelativistic physics is that the scalar product is invariant to velocity shifts in a nonrelativistic development but a universal and finite speed of light precludes velocity shift invariance in relativistic developments. The elevation of $\mathbf{x}$ is Hermitian in nonrelativistic but not relativistic physics as a consequence.[q] The elevation of location is the Hermitian location operator in nonrelativistic physics, but a

[q] As a consequence of $\langle e^{i\mathbf{q}\cdot\mathbf{x}} f(x) | e^{i\mathbf{q}\cdot\mathbf{x}} g(x)\rangle = \langle f(x)|g(x)\rangle$ for the $\mathcal{L}^2$ scalar product applicable in nonrelativistic physics, the generator of velocity shifts, the quantization of $\mathbf{x}$, is Hermitian. The scalar product in relativistic physics uses a Källén-Lehmann form two-point function [9] and then $e^{i\mathbf{q}\cdot\mathbf{x}}$ is not Hermitian.

Hermitian elevation of location is precluded by relativity even for free fields, [43, 66] and section 5. Yet, clearly, location remains an observable.

The quantization of location, also designated the elevation of location, is the Hilbert space operator with eigenfunctions that are Dirac delta functions over space. Every point in $\mathbb{R}^3$ is an eigenvalue of the quantization of location and the eigenfunctions are generalized eigenfunctions. The number of orthogonal generalized eigenfunctions of location is uncountable and these eigenfunctions can therefore not be elements of a separable rigged Hilbert space, section 5. The point support of a Dirac delta function corresponds with the classical concept of location as a point in $\mathbb{R}^3$. The Hermitian Hilbert space operator that corresponds to location is distinct from the canonical quantization of location. Quantizations are not necessarily Hermitian. From [13, 60] and appendix 6.2.6, to correspond to a classical dynamical variable, that is, to have a real expected value for all state descriptions, a Hilbert space operator must be Hermitian. However, the relayivistic Hilbert space scalar product is not compatible with eigenfunctions that are delta functions. Eigenfunctions of a Hermitian Hilbert space operator with distinct eigenvalues are necessarily orthogonal in the Hilbert space scalar product [3]. A relativistic scalar product has a Källén-Lehmann form two-point function to achieve the physically necessary properties of Poincaré invariance and positive energies. Dirac delta functions are not orthogonal in this scalar product.

$$\langle f(x_0)\delta(\mathbf{x}-\mathbf{y}_1)|f(x_0)\delta(\mathbf{x}-\mathbf{y}_2)\rangle \neq 0 \qquad \text{when} \quad \mathbf{y}_1 \neq \mathbf{y}_2.$$

Hence, Dirac delta functions over space can not be the eigenfunctions of a Hermitian operator in relativistic physics [43, 66].

With the revised, approximate and conditional quantum-classical correspondence, there are Hermitian operators $\widehat{X}_\nu$ that correspond to location in relativistic physics. The eigenfunctions of these operators are Theodore Newton and Eugene Wigner's relativistically invariant localized functions [43]. In the momentum domain, the relativistically invariant localized functions $\psi_x(x_1)$ [43] are

$$\tilde{\psi}_x(p) = (2\omega)^{\frac{1}{2}}\ e^{-ipx} \tag{211}$$

labeled by spacetime points $x \in \mathbb{R}^4$. These functions are generalized eigenfunctions of a Hermitian operator defined in the Hilbert space for a relativistic free field. There is an eigenfunction with eigenvalue $\mathbf{x}$ for every location $\mathbf{x} \in \mathbb{R}^3$. For nonrelativistic momenta, $\tilde{\psi}_x(p) \approx (2m)^{\frac{1}{2}}\ e^{-ipx}$ and in this sense the relativistically invariant localized functions approximate Dirac delta functions. For the constructions, one-argument functions always describe a single elementary particle and the $\widehat{X}_\nu$ are location operators for that elementary particle in the one-particle subspace. In subspaces with a greater number of arguments and with relativistic momenta in a multiple species construction, the $\widehat{X}_\nu$ remain location operators but the location is not associated with a determined number nor species of particles. These functions are also discussed in appendix 6.3.

Hermitian location operators $\widehat{X}_\nu$ are described by spectral theory for rigged Hilbert space operators (theorem 1, appendix to section 4 [21], lemma 5.6.7 [40], and chapters 7-10 [24]). The three Hermitian location operators are

$$\widehat{X}_\nu := \int_{\mathbb{R}^3} d\mathbf{x}\; x_\nu E_x. \tag{212}$$

$E_x \sim |\psi_x\rangle\langle\psi_x|$ projects one-argument functions in $\mathbf{H}_\mathcal{P}$ to the relativistically invariant localized functions located near $\mathbf{x}$ at time $x_0 = ct$ (211). $\nu = 1, 2, 3$, $x = x_0, \mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^3$ and x_ν are the three Cartesian components of $\mathbf{x}$. The relativistically invariant localized functions are mutually orthogonal, $\langle\psi_{y_1}|\psi_{y_2}\rangle = 0$ if $\mathbf{y}_1 \neq \mathbf{y}_2$ and $y_{10} = y_{20}$. The relativistically invariant localized functions are generalized eigenfunctions of the $\widehat{X}_\nu$.

$$\widehat{X}_\nu \psi_x(x_1) = x_\nu \psi_x(x_1),$$

and

$$Q_\chi := \int_\chi d\mathbf{x}\; E_x$$

are projection operators that provide a resolution of unity. $d\mathbf{x}$ is Lebesgue measure on $\mathbb{R}^3$, and the χ refer to measurable volumes within $\mathbb{R}^3$. From the orthogonality of the $\psi_x(x_1)$, $E_x E_{x'} = 0$ if $x_0 = x'_0$ and $\mathbf{x} \neq \mathbf{x}'$. The Q_χ are projections (idempotent, self-adjoint operators)[r] in the one particle subspace of $\mathbf{H}_\mathcal{P}$. The functions $\psi_x(x_1)$ are essentially localized but not strictly localized. The ranges of the projections Q_χ are subspaces of positive energy, Poincaré covariant, essentially localized states and orthogonal projection operators are Birkhoff and von Neumann's experimental propositions [6, 60].

There are many first order linear differential operators that canonically commute with the Hermitian momentum operators P_ν. Examples are constructed in appendix 6.3 and include: the Hermitian quantization X_ν of $\mathbf{x}$ applicable in $\mathcal{L}^2$ Hilbert spaces of nonrelativistic physics; the Hermitian operator $\widehat{X}_\nu$ that has Newton and Wigner's relativistically invariant localized functions [43] as eigenfunctions and applies in Fock space developments of the relativistic free field; and the Hermitian operator that applies for the section 3 construction of a Hilbert space based on $\underline{\mathcal{P}}$.

For a free field, the Hermitian location operator $\widehat{X}_\nu$ in (212) that applies in the one-particle subspace extends to multiple argument states by second quantization. If the interaction of states is significant then multiple-argument components of states are not necessarily interpretable as

[r] $E_x E_{x'} = 0$ if $\mathbf{x} \neq \mathbf{x}'$ and $t = t'$, and then $Q_\chi Q_{\chi'} = 0$ if $\chi \cap \chi' = \emptyset$ for spatial volumes χ, χ' within a constant time plane. The functions $\psi_x(x_1)$ and $\psi_{x'}(x_1)$ are orthogonal at coincident times. However, evaluated at distinct times $t \neq t'$, $E_x E_{x'} \neq 0$. If there were $\mathcal{Q}_\Delta$ such that $\mathcal{Q}_\Delta \mathcal{Q}_{\Delta'} = 0$ for projections onto spacetime volumes $\Delta, \Delta' \subset \mathbb{R}^4$ with $\Delta \cap \Delta' = \emptyset$ and space-like separated, then those $\mathcal{Q}_\Delta = 0$ [66]. There are no such projections. The volumes with $Q_\chi Q_{\chi'} = 0$ are insufficient to conclude from Bogolubov's edge of the wedge theorem that $Q_\chi = 0$.

determined numbers or species of particles, section 3.1. The number and species of particles and even whether the state is perceived as particles is not evident unless interaction is negligible for the state of interest. That is, a free field location operator necessarily applies approximately only for states that are well represented by classical particles. Free field and nonrelativistic states are readily interpreted as consisting of particular numbers and species of particle. Otherwise, the dominant supports of multiple argument functions correspond to likely locations but the numbers and species of particles at those locations is indeterminant.

Relativistic location illustrates a key difference between the less constrained, alternative development and a canonical quantization: the distinction between an elevation of a classical dynamical variable and a Hermitian operator that corresponds to the classical dynamical variable. For location, three quantities are distinguished: $\mathbf{x}$, X_ν, and $\widehat{X}_\nu$. The vector $\mathbf{x} \in \mathbb{R}^3$ specifies a classical location and a point in the domain of the state describing functions. The X_ν for $\nu = 1, 2, 3$ are elevations of the three components of $\mathbf{x}$, operators with Dirac delta functions as generalized eigenfunctions. And, the $\widehat{X}_\nu$ are Hermitian Hilbert space operators with generalized eigenfunctions that are the relativistically invariant localized functions of Newton and Wigner [43]. These generalized eigenfunctions describe the natural generalization of localized states in a relativistic development. The "elevation of c-number to q-number" conjecture is that the X_ν should equal $\widehat{X}_\nu$. The contradiction to the Hermiticity of X_ν in relativistic physics, included in section 5, is a "localization problem" of RQFT. The relativistically invariant localized functions conditionally approximate Dirac delta functions. That a small neighborhood of a location $\mathbf{x}$ includes all perceptions of location to arbitrarily great likelihood suffices physically as the description of localized. Location in relativistic physics is also discussed in [43, 66], section 3.1, and appendix 6.3. The "localization problem" as well as the lack of nontrivial realizations in relativistic quantum physics are overcome by adopting more appropriate relativistic quantum-classical correspondences.

With consideration limited to observation of features associated with classical bodies, if we calculate that the bodies move guided by fields $A(x)_\kappa$ with sources on the bodies (e.g., the Lorentz force and Maxwell's equations in electrodynamics, or geometrodynamical gravity) and this classical dynamics produces an accurate approximation to the quantum dynamics when the supports of state describing functions are well represented by classical bodies, then we would say that the quantum model corresponds with the classical field theory. This correspondence is valid regardless of whether there is a "quantization" of the classical fields. A correspondence of classical field theory with quantum dynamics is established by the approximation of the motions of observed bodies.

6.5 Inconsistency of the classical description with nature

Quantum mechanics is a striking change from classical descriptions. Several competing schools of thought persist on whether quantum mechanical descriptions actually depict nature. The difficulty for many is that quantum mechanics forces us to abandon well-developed classical

intuition.

The considerations listed below require a general abandonment of classical concepts. The conflict of classical concepts with nature is illustrated by many considerations including:

1. Planck's calculation of the spectrum of black body radiation
2. the heat capacity of solids
3. the photoelectric effect
4. Stokes fluorescence and Compton scattering
5. Gibb's paradox
6. the creation and annihilation of particles
7. interference patterns in a Michelson interferometer at very low "single quantum at a time" light intensity
8. discrete energy bands in the radiation spectra from atoms
9. the Einstein-Podolsky-Rosen paradox for conserved quantities such as angular momentum, discussed in appendix 6.6.

Taken together, these considerations do not rectify with a classical world view. Each is a motivation for quantum mechanics, and together with principles of simplicity and universality, a quantum mechanical description for nature is indicated.

For consistency with observations, Max Planck's calculation for the spectrum of black body radiation included that the energy in electromagnetic radiation is quantized in discrete particle-like amounts with an energy E proportional to photon frequency ν, $E = h\nu$. h became known as Planck's constant. Planck's revelation and Albert Einstein's insight that the photoelectric effect is also explained by a quantized photon energy produced agreement with observation. The observed photoelectric effect is strong evidence for the interpretation of electromagnetic radiation as photons. In the photoelectric effect, electrons are not emitted from a surface until the light frequency is sufficient, that is, until the light quanta include sufficient energy that the dominant reaction, interaction of an electron with a single photon, results in an electron with sufficient energy to escape the surface. This observation is (nearly) independent of the amplitude of the incident radiation. The exception is nonlinear optics, if the electron absorbs the energy of multiple photons. These results contradict the classical description of electromagnetism as waves and energy as a freely specified real parameter. If the illumination is a wave, we should anticipate that once the amplitude of the wave was sufficient that electrons would escape the surface. In his identification of the quantized energy $E = h\nu$ of an electromagnetic field, Einstein was also motivated by the observed heat capacity of solids, in particular, contradictions

to the equipartition of energy in classical statistical mechanics, and Stokes fluorescence [55], that the frequency of a photon emitted by a body was less than the illuminating frequency. (Stokes fluorescence has since been supplemented with observations of anti-Stokes radiation, when vibrational modes of an emitter contribute to the energy of the re-emitted photon.) Together with Joseph John Thomson's observations of the electron, Robert Milliken and Harvey Fletcher's measurement of a discrete electric charge, and Jean Perrin's observations and Albert Einstein's description of Brownian motion, these findings provided motivation for the descripion of matter as a composition of atoms.

The indistinguishability of bodies of the same mass, spin, polarization and charges naturally resolves Gibb's paradox. Gibb's paradox arises if entropy is not an extensive quantity. In statistical physics, an extensive quantity is proportional to the amount of substance, that is, for an extensive entropy, twice the volume of gas should have twice the entropy. Were entropy not extensive, then Gibbs paradox is a violation of the second law of thermodynamics, that entropy of isolated systems does not decrease. Other solutions have been suggested to resolve Gibb's paradox, but the indistinguishability of bodies is a natural resolution consistent with the Bose-Einstein or Fermi-Dirac statistics of similarly described bodies in quantum mechanics. Note that this indistinguishability is in contradiction to the classical concept that the trajectories of individual bodies can be distinguished and followed.

The creation and annihilation of particles is another contradiction to the classical concept of continuous trajectories. In these cases, trajectories disappear and trajectories with distinct descriptions appear. Such transformations of identifiable bodies are not described by classical physics. With creation and annihilation, one cannot follow the trajectory of a single identifiable entity, nor generally even determine the number of bodies in a relativistic state description.

A Michelson interferometer consists of two light paths split and later recombined using a half-silvered mirror, two reflecting mirrors, and an optical path matching slab of clear glass. An interference pattern of light and dark intensity rings is visible on detectors at an end of the optical path, for example, on photographic film. This pattern persists even as the intensity of the light is lowered. From the model of light quanta established in the photoelectric effect and Planck's calculation of black body radiation spectra, it follows that the illumination can be reduced to less than a single photon at a time within the interferometer on average. This establishes that each photon interferes with itself, and since the arms of the interferometer are separated and can be of unequal length, that the photon takes both paths. This contradicts a classical description of the photon as a classical particle with a definite trajectory. The interference pattern is consistent with the classical concept of electromagnetism as a wave, but that description is contradicted by the photoelectric effect. The interferometer is a strong argument for the reality of the quantum description of state. It is difficult to understand how at low intensities the interference pattern can be responsive to changes in arm path length without a physical presence in each of the two separate arms for each photon.

Discrete line spectra observed in emissions of light, for example, from a gas of hydrogen or sodium atoms, contradict that energy is a real number parameter, an initial condition for

classical equations of motion. Line spectra contradict that any in a range of real numbers can provide the initial conditions that result in a hydrogen atom. There is no mechanism within classical descriptions that results in the observed atomic spectra (nor the observed heat capacities of solids at low temperatures). One of the predictions of nonrelativistic quantum mechanics is line spectra, and observations of the Lamb shift of hydrogen atom energy levels is one of the most precise tests in physics. The Lamb shift is correctly estimated by the Feynman series rules for quantum electrodynamics.

Quantum mechanics resolves many observed flaws of classical descriptions, from extensive entropy to conservation laws for quantized quantities to discrete atomic spectra to the consistency of observed interference patterns with particle-like descriptions of waves. Despite all this, classical descriptions of physics continue to be used as the underpinnings of quantum physics. The concept of a continuous evolution of distinguishable bodies traveling trajectories, the descriptions of Newtonian physics and Einstein's geometrodynamics, are evidently useful approximations to quantum physics but in limited instances. The concept that quantum dynamics is the "quantization" of these classical dynamic descriptions rests on expedience, experience with nonrelativistic quantum mechanics, and successful but limited phenomenology. Any correspondence of quantum with classical need only occur in appropriate instances, and canonical quantization is an unjustified extrapolation otherwise.

6.6 The Einstein-Podolsky-Rosen paradox

The Einstein-Podolsky-Rosen (EPR) [16] paradox and confirmation of Bell's inequalities [5] illustrate that a classical description is inconsistent with nature. The quantum mechanical description of nature has been described as "bizarre" but, this description reflects that quantum mechanics demands rejection of long established classical concepts. The EPR paradox [16] illustrates that the quantum mechanical description of nature contradicts classical concepts. Einstein, Podolsky and Rosen develop the argument that quantum mechanics must be incomplete because the quantum description conflicts with a classical concept of state. Of course, the alternative is that the classical description is inconsistent with nature. To develop the EPR paradox, it is observed that spin angular momentum is quantized with the same discrete values on any axis of observation. The paradox arises if a spin zero particle decays into two spin one-half particles that subsequently fly apart. Sufficiently separated, we can determine the spin state of an arbitrarily distant particle by observing the paired particle: the distant spin is the one that paired with the near spin conserves angular momentum. The paradox is that we determine the spin of the non-causally related distant particle differently as determined by our selection of measurement axis. If the distant particle is classically described, then it has a determined spin unaffected by our observation of the nearby particle. The resolution to the conflict is entanglement, a concept in quantum mechanics that is not supported in a classical description. That is, the resolution of the EPR paradox is not a contradiction to quantum mechanics but a contradiction to classical concepts, to the classical description of state.

Entanglement is natural in the Hilbert space description of multiple particle states [50]. The paradox is resolved by the realization that a classical description is not consistent with nature. Although Einstein motivated quantum mechanics with studies of the photoelectric effect and the heat capacity of solids [55], he is perhaps most celebrated for his understanding of time and motion described in special relativity and geometrodynamical gravity. Einstein's derivation of gravity relies on classical description of nature. Expressed in the EPR paradox, Einstein did not embrace implications of quantum mechanics. I presume that Einstein was persuaded by his immensely successful insights into time and motion using classical concepts that are in conflict with quantum mechanics. Also, more consistent perspectives on quantum mechanics [11] developed later. The principle of equivalence, that acceleration is equivalent to a gravitational force, relies on the classical description of a body characterized by a trajectory. In a quantum mechanical description, the characterization of the velocity of a body deteriorates with enhancements in location accuracy: both location and velocity are never known with a precision that exceeds the Heisenberg uncertainty bound. We cannot arbitrarily precisely associate the acceleration of a body with its locations.[s] It is a great irony that just as problematic aspects of Newtonian mechanics were resolved with dynamical, relativistic time and the equivalence of acceleration with gravitation, atomic physics necessitated the new mechanics that supersedes classical developments.

Flaws in classical physics such as a non-causal radiation reaction force in classical electrodynamics [29] can be tolerated since classical descriptions only approximate the more fundamental, and causal, quantum descriptions. Indicated in Feynman's quote in section 4, it could be anticipated that the transition from the well established and successful practice of classical physics to quantum mechanics would be slow. Even though it was observation that necessitated the development of the new mechanics, adoption of quantum mechanics is inhibited by the established reliance on classical concepts. Quantum mechanics is the more comprehensive and unified model. Although the classical perspective is an accurate approximation of the quantum description in our common experience, the classical perspective is contradicted by nature.

6.7 Heisenberg uncertainty

One of the great insights in the development of quantum mechanics is that location and momentum are not independently specified descriptions of bodies. Both location and momenta derive from a state describing function $\psi(x)$. The supports of functions $\psi(x)$ are associated with the likelihoods of locations and the supports of the Fourier transforms $\tilde{\psi}(p)$ are associated

[s] Although for typical nonrelativistic Hamiltonians acceleration and location commute, $[X_\nu, [H, P_\nu]] = 0$, an estimate for acceleration from a sequence of localizing measurements fails due to the lack of a trajectory precisely associated with a quantum description. Each observation conveying location knowledge contributes velocity uncertainty. There are no classical state descriptions included in quantum mechanics. For many states, particularly state descriptions with high energies and overlapping supports, the principle of equivalence does not even approximate the evolution for the state describing functions.

with the likelihoods of momenta. Likelihoods are provided by Born's rule from the Hilbert space scalar product. Mean values of location and momentum are independently specified but variances are constrained as described by the Heisenberg principle. The Heisenberg uncertainty principle follows from the quantum mechanical description of states. If the scalar product is $\mathcal{L}^2$, appropriate in nonrelativistic physics, then the location operator corresponds to the Hermitian operator realized by multiplication of the function $\psi(x)$ by the value of an argument $\mathbf{x}_\nu$ and the momentum operator corresponds to the Hermitian operator that is the similarly quantized multiplication of $\tilde{\psi}(p)$ by $\mathbf{p}_\nu$ in the momentum domain. Using the Fourier transform relations (18) and linearity, in $\mathcal{L}^2$ location corresponds to $-i\hbar d/dp_\nu$ and momentum to $i\hbar d/dx_\nu$. This is discussed further in appendix 6.3. If at a time t the support of $\psi(x)$ is dominantly supported in the neighborhood of a point $\mathbf{y}(t)$, then

$$\langle\psi|X_\nu\psi\rangle := \int d\mathbf{x}\ \mathbf{x}_\nu|\psi(x)|^2 \approx \mathbf{y}_\nu(t)\int d\mathbf{x}\ |\psi(x)|^2 = \mathbf{y}_\nu(t)$$

for a normalized $\psi(x)$. $\mathbf{y}(t)$ is any suitable representative of the neighborhood of support. From Parseval's equality (17), the definition of Fourier transform (16) and implied identities, if at a time t the support of the Fourier transform $\tilde{\psi}(\mathbf{p},t)$ is dominantly supported near a momentum $\mathbf{q}(t) = \hbar\mathbf{p}(t)$, then with $\nu = 1,2,3$,

$$\langle\psi|P_\nu\psi\rangle := \hbar\int d\mathbf{p}\ \mathbf{p}_\nu|\tilde{\psi}(\mathbf{p},t)|^2 = i\hbar\int d\mathbf{x}\ \overline{\psi(\mathbf{x},t)}\ \frac{d\psi(\mathbf{x},t)}{dx_\nu} \approx \mathbf{q}_\nu(t).$$

There is no knowledge of momentum for a location eigenfunction, and no knowledge of location for a momentum eigenfunction. Are there classical-like selections for functions that describe quantum mechanical states? Here, a nearly classical state is one with both location and velocity known precisely. For finite masses, the time derivative of location is velocity $\mathbf{v}$ and in the nonrelativistic approximation $\mathbf{p} \approx m\mathbf{v}$. There is no bound on the precision of knowledge of the mass. Are there states that approximate both location and momentum arbitrarily well? The result is that there is an optimal accuracy that can be simultaneously obtained for knowledge of location and momentum, and a choice of functions that provides the optimal simultaneous knowledge. This result is the *Heisenberg uncertainty principle*. In quantum mechanics, not only is dynamics no longer described by a smooth trajectory specified by an initial location and velocity, but we cannot know both the location and the time derivative of the location of a body sufficiently well to specify a trajectory. That is, our description of state does not support the concept of Newtonian mechanics. The Heisenberg uncertainty principle is a lower bound on the breadth of the supports in location and momentum in a quantum mechanical description. We can approximate the location and time derivative of location for a body, and this approximation becomes better the heavier the body, but the concept of Newtonian mechanics is not supported by the quantum description of state. Also, in quantum mechanics, we cannot necessarily identify a particular body to propagate forward in time. Similarly described bodies

are indistinguishable and cannot be labeled to identify a trajectory except while a single body is sufficiently isolated from other bodies to reliably identify the body. While isolated, a dominant volume in the support of a function that describes a state can be identified and followed with a high likelihood until it approaches other similar bodies. At this point, its identity becomes lost in ambiguity.

The Heisenberg uncertainty principle is derived in ordinary, nonrelativistic quantum mechanics. It is a result that follows from the $\mathcal{L}^2$ scalar product applicable in nonrelativistic physics. In $\mathcal{L}^2$, three location operators $X_\nu = x_\nu$ and three momentum operators $P_\nu = i\hbar d/dx_\nu$, one for each spatial dimension, are self-adjoint in $\mathcal{L}^2$ and satisfy the Born-Heisenberg-Jordan relation

$$[X_\nu, P_\nu] = -i\hbar, \tag{213}$$

a canonical commutation relation (CCR). For an arbitrary state $|\psi\rangle$ labeled by a function $\psi(x) \in \mathcal{L}^2$ in the intersection of the domains of X_ν and P_ν (with convergent $|\langle\psi|X_\nu\psi\rangle|$ and $|\langle\psi|P_\nu\psi\rangle|$), define operators

$$A := X_\nu - \langle\psi|X_\nu\psi\rangle, \quad \text{and} \qquad B := P_\nu - \langle\psi|P_\nu\psi\rangle.$$

$\nu = 1, 2$ or 3 and the intersection of the domains includes states labeled by the Schwartz tempered functions that are dense in $\mathcal{L}^2$. From the commutation of X_ν and P_ν it follows that $[A, B] = -i\hbar$ and that A, B are mean zero for the state $|\psi\rangle$. From the interpretation of $\langle\psi|A\psi\rangle$ as the mean value of the quantity associated with the operator A for normalized states $\langle\psi|\psi\rangle = 1$, identify variances of location and momenta as

$$\sigma_x^2 = \langle\psi|A^2\psi\rangle, \quad \text{and} \qquad \sigma_p^2 = \langle\psi|B^2\psi\rangle.$$

Self-adjointness of A and B follows from the self-adjointness of X_ν and P_ν on $\mathcal{L}^2$. Self-adjointness and the Cauchy-Schwarz-Bunyakovsky inequality then provide that

$$|\langle\psi|AB\psi\rangle|^2 = |\langle A\psi|B\psi\rangle|^2 \leq \langle A\psi|A\psi\rangle \langle B\psi|B\psi\rangle = \langle\psi|A^2\psi\rangle \langle\psi|B^2\psi\rangle = \sigma_x^2\sigma_p^2.$$

The lower bound, equality, is achieved if $|A\psi\rangle = c|B\psi\rangle$ for a complex constant c. From the definition of commutator, linearity of scalar products, and the property of scalar products that $\langle u|v\rangle = \overline{\langle v|u\rangle}$, identify that

$$\langle\psi|[A, B]\psi\rangle = \langle\psi|AB\psi\rangle - \langle\psi|BA\psi\rangle = \langle\psi|AB\psi\rangle - \langle AB\psi|\psi\rangle = 2i\Im m(\langle\psi|AB\psi\rangle),$$

twice the imaginary part. The imaginary part of z has a magnitude bounded by the magnitude of z. This bound and the commutation relation for A and B result in that

$$|\langle\psi|[A, B]\psi\rangle|^2 = |i\hbar\langle\psi|\psi\rangle|^2 = \hbar^2 = |2i\Im m(\langle\psi|AB\psi\rangle)|^2 \leq 4|\langle\psi|AB\psi\rangle|^2 \leq 4\sigma_x^2\sigma_p^2.$$

This is Heisenberg's uncertainty principle. For states in $\mathcal{L}^2$,

$$\sigma_x\sigma_p \geq \hbar/2,$$

a lower limit on the geometric mean of the variances of location and momentum. This limits the simultaneous accuracy of location and momentum descriptions in each dimension ν. The lower bound is achieved if $\langle\psi|AB\psi\rangle$ is imaginary and $|A\psi\rangle = c|B\psi\rangle$ for $c \in \mathbb{C}$. The labels $\psi(x)$ with $x \in \mathbb{R}^3$ for the states that meet both lower bounds are Gaussian functions (97),

$$\psi(x) = \frac{e^{-(x-x_o)^2/(4L^2)+ip_o x/\hbar}}{(2\pi L^2)^{\frac{1}{4}}}.$$

These Gaussian functions satisfy

$$\langle\psi|AB\psi\rangle = i\int_{\mathbb{R}^3} dx\ \frac{(x-x_o)^2}{2L^2}|\psi(x)|^2$$

that is imaginary and

$$A\psi(x) = (x-x_o)\psi(x) = cB\psi(x) = c(-i\hbar\frac{d\psi(x)}{dx} - p_o\psi(x)) = c\ \frac{i\hbar(x-x_o)}{2L^2}\psi(x)$$

for real $x_o := \langle\psi|X_\nu\psi\rangle$ and $p_o := \langle\psi|P_\nu\psi\rangle$, $c = 2L^2/(i\hbar)$ and $\langle\psi|\psi\rangle = 1$. The parameter $L = \sigma_x$ and $\sigma_p = \hbar/(2L)$. This is the function shape in each dimension. The spatial function that labels a minimum uncertainty state is the product of three factors, one for each spatial dimension. These Gaussian functions are denoted minimum packets and are the states most classical-like states in the sense that the geometric mean of the uncertainties in simultaneous knowledge of the location and momentum of the state is minimized.

The Heisenberg uncertainty principle applies in relativistic physics with some revision. In a relativistic development, the operators X_ν ($|X_\nu\psi\rangle := |x_\nu\psi\rangle$) are not self-adjoint and therefore are not the quantization of location. As a consequence, the development above of the Heisenberg uncertainty relation does not apply for operator pairs X_ν, P_ν. Nevertheless, the Heisenberg uncertainty principle applies when classical approximations to the relativistic physics apply, and with X_ν replaced by the Hermitian $\widehat{X}_\nu := -i\hbar\omega^{1/2}d/dp_\nu\omega^{-1/2}$, the relativistic, single body location operator [43]. Classical particle approximations apply when states are described by functions with isolated (105) concentrations of support well represented by a single location (102) and momentum (103). The eigenfunctions of $\widehat{X}_\nu$ are the relativistically invariant localized functions. If the $\widehat{X}_\nu$ is used in the development of the Heisenberg uncertainty relation, then the minimum uncertainty packets are not Gaussian functions. The relativistic minimum uncertainty packets are normalized inverse Fourier transforms of

$$\tilde{\psi}(\mathbf{p}) = \omega^{1/2}e^{-\sigma_o^2(\mathbf{p}-\mathbf{w})^2}e^{i\mathbf{p}\cdot\mathbf{x}_o/\hbar}.$$

If the state description consists of sufficiently isolated concentrations in the support of each argument and these supports are well represented by single locations and momenta, then the

state can be represented by classical particles and the Heisenberg uncertainty principle applies for functions $\psi(\mathbf{x})$ used to define functions in $\mathbf{H}_{\mathcal{P}}$.

$$\tilde{\varphi}(p) := \frac{(p_0+\omega)\tilde{\psi}(\mathbf{p})}{\sqrt{2\omega}} \in \mathbf{H}_{\mathcal{P}}. \tag{214}$$

Due to cluster decomposition, the contributions to scalar products of the high order connected functions are negligible when a state consists solely of widely spatially isolated bodies. In these cases, the $\mathcal{W}_{k,n-k}$ are well approximated by the free field contribution (44) to the VEV. From (44), the significant contributions to scalar products (21) for product states f_n,

$$f_n((x)_n) = \prod_{j=1}^{n} \varphi_j(x_j) \in \mathbf{H}_{\mathcal{P}}$$

and functions φ_j with Fourier transforms of the form (214) consists of sums of products of factors $W_2(\varphi_j^* \varphi_\ell)$. Substitution results in

$$\begin{aligned} W_2(\varphi_j^*\varphi_\ell) &= \int dp_1 dp_2 \; \tilde{W}_2(p_1,p_2)\overline{\tilde{\varphi}_j(-p_1)}\tilde{\varphi}_\ell(p_2) \\ &= \int dp_1 dp_2 \; \delta(\mathbf{p}_1+\mathbf{p}_2)\, \frac{\delta(p_{01}+\omega_1)}{\sqrt{2\omega_1}}\, \frac{\delta(p_{02}-\omega_2)}{\sqrt{2\omega_2}}\, \overline{\tilde{\varphi}_j(-p_1)}\tilde{\varphi}_\ell(p_2) \\ &= \int d\mathbf{p}_2 \; \overline{\tilde{\psi}_j(\mathbf{p}_2)}\tilde{\psi}_\ell(\mathbf{p}_2) \\ &= \int d\mathbf{x} \; \overline{\psi_j(\mathbf{x})}\psi_\ell(\mathbf{x}). \end{aligned} \tag{215}$$

This is the $\mathcal{L}^2$ scalar product of the $\psi_j(\mathbf{x})$. The Heisenberg uncertainty bound applies to the breadths of location and momentum support of $W_2(\varphi_j^*\varphi_j)$ with $\psi_j(\mathbf{x})$ substituted for $\varphi_j(x)$.

Gaussian functions provide an explicit example of an expansion of a state description as a linear combination of more localized Gaussian elements in the Hilbert space. The identity

$$e^{-\frac{x^2}{4L_0^2}} = \sqrt{\frac{1}{\epsilon^2(1-\epsilon^2)4L_0^2\pi}} \int ds \; e^{-\frac{s^2}{4(1-\epsilon^2)L_0^2}} e^{-\frac{(x-s)^2}{4\epsilon^2 L_0^2}}$$

with $0 < \epsilon \ll 1$ is an expansion of the more broadly supported Gaussian function as a linear combination of arbitrarily narrower Gaussian functions centered on s. This expansion is in one dimension. The likelihood of a transition from the normalized state characterized by

$$\psi(x) = N_\psi e^{-\frac{x^2}{4L_0^2}}$$

to the more localized, normalized state characterized by

$$g_s(x) = N_s e^{-\frac{(x-s)^2}{4\epsilon^2 L_0^2}}$$

is

$$|\langle g_s|\psi\rangle|^2 = \frac{2\epsilon}{1+\epsilon^2}\, e^{-\frac{s^2}{4(1+\epsilon^2)L_0^2}}$$

with $\langle\psi|\psi\rangle = \langle g_s|g_s\rangle = 1$. As anticipated, the likelihood of transition to a function centered on s declines rapidly the farther s is from the center of support at $x = 0$. And, the likelihood of transition to a function of width $\epsilon\sigma$ declines slowly with reductions in relative function spread.

6.8 Feynman-Dyson series and the constructions

A renormalized perturbative series for scattering amplitudes [23, 52, 61] is discussed in this appendix to contrast Feynman (also refered to as Feynman-Dyson, or Dyson) series scattering amplitudes with the explicit scattering amplitudes from the constructions in section 3. In this appendix, the comparison of quantum mechanical constructions with Feynman series is illustrated for the example of one neutral scalar field $\Phi(x)$, $N_c = 1$. Feynman series apply only to scattering [8], to infinite interval transition amplitudes.

From section 3.9, the constructed scattering amplitudes are infinite interval limits of more general state transition amplitudes. Appropriate states are interpretable as classical bodies described by momenta using plane wave limits of localized states. For states (2) described by product functions of point support at times λ_j and with momentum support centered on $\mathbf{q}_j$,

$$f_n((x)_n) = \prod_{j=1}^{n} \ell(x_j; \lambda_j, \mathbf{q}_j)$$

[33, 35, 37], a transition from m to n freely propagating particles is described by the LSZ (Lehmann-Symanzik-Zimmermann) expressions for scattering amplitudes [9],

$$\begin{aligned} S_{n.m} &= \lim_{\lambda\to\infty} \langle U(\lambda)\ell(\lambda, \mathbf{q}_{m+1}) \dots \ell(\lambda, \mathbf{q}_{m+n})|U(\text{–}\lambda)\ell(\text{–}\lambda, \mathbf{q}_1) \dots \ell(\text{–}\lambda, \mathbf{q}_m)\rangle \\ &= \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+1})) \dots \Phi(\ell(\lambda, \mathbf{q}_{m+n}))\Omega|U(\text{–}\lambda)\Phi(\ell(\text{–}\lambda, \mathbf{q}_1)) \dots \Phi(\ell(\text{–}\lambda, \mathbf{q}_m))\Omega\rangle \end{aligned} \tag{216}$$

and

$$S_{n.m} = \lim_{\lambda\to\infty} \langle U(\lambda)\Phi(\ell(\lambda, \mathbf{q}_{m+n}))U(\lambda)^{-1} \dots U(\lambda)\Omega|U(\text{–}\lambda)\Phi(\ell(\text{–}\lambda, \mathbf{q}_1))U(\text{–}\lambda)^{-1} \dots U(\text{–}\lambda)\Omega\rangle$$

with $U(\lambda)$ the unitary time translation operator, and applying the definition of field (2) with the notation of section 3.1.3. The state describing functions $\ell(x_j; \lambda_j, \mathbf{q}_j)$ are described in (96) in section 3.9.

$$\tilde{\ell}(p_j; \lambda_j, \mathbf{q}_j) := e^{iE_j\lambda_j}(\omega_j + E_j)\tilde{f}(\mathbf{p}_j - \mathbf{q}_j)$$

with $\lambda_j \in \mathbb{R}$, $\mathbf{q}_j \in \mathbb{R}^3$ and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. $\ell(x_j; \lambda, \mathbf{q}_j)$ is within the completion $\mathbf{H}_\mathcal{P}$ of the $\mathcal{P}(\mathbb{R}^4)$ described in section 3.7, Ω designates the vacuum and the VEV are constructed in section

3. $\mathcal{S}(\mathbb{R}^3)$ includes functions with Fourier transforms that are delta sequences with supports concentrated near the momenta $\mathbf{q}_j$. A convenient choice for function $\tilde{f}(\mathbf{p})$ is the Gaussian function (97), a point-wise nonnegative delta sequence. The cluster decomposition axiom A.6 provides that the Gaussian functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are described at large times as free particles if the functions $\ell(x_j; \lambda, \mathbf{q}_j)$ are translated to center the spatial support on free particle trajectories with distinct momenta $\mathbf{q}_j$.

The LSZ scattering amplitudes (216) are VEV of products of temporal translations of fields $\Phi(\ell(t, \mathbf{q}))$. From section 3.9 and with $\lambda_j := \lambda$, temporal translations of these fields are independent of time.

$$U(t)\Phi(\ell(t, \mathbf{q}))U(t)^{-1} = \Phi(\ell(0, \mathbf{q}))$$

due to the limitation of the spectral support of VEV from section 3 to mass shells and the selection of compensating phases in $\ell(x_j; \lambda_j, \mathbf{q}_j)$. In the constructions, the scattering amplitudes (216) are fully quantum mechanical transition amplitudes defined for any state and including finite time differences.

The RQFT [7, 23, 52, 61] rule for calculation of plane wave scattering amplitudes is to evaluate the generalized functions

$$S_{n,m}((p)_{n+m}) := \langle \widetilde{\Phi_o^+}(p_n) \dots \widetilde{\Phi_o^+}(p_1)\Omega_o | U_D(t, -t) \widetilde{\Phi_o^+}(p_{n+1}) \dots \widetilde{\Phi_o^+}(p_{n+m})\Omega_o \rangle \tag{217}$$

for $t = \infty$. These scattering amplitudes are expanded in free field VEV. Free field operators are distinguished in this appendix by the notation $\Phi_o(x)$. The Dyson operator $U_D(t_1, t_2)$ is provided below in (222). Creation components $\Phi_o^+(x)$ of free fields are introduced in section 3.3.1. The scattering amplitudes $S_{n,m}((p)_{n+m})$ apply the Fourier transform of generalized functions [19],

$$\Phi_o^+(\underline{f}) = \widetilde{\Phi_o^+}(\underline{\tilde{f}}). \tag{218}$$

Introduced in section 3.3.1, the free field

$$\Phi_o := \Phi_o^+ + \Phi_o^-$$

has a cyclic vacuum state Ω_o and an annihilation component $\Phi_o^- \, \Omega_o = 0$. From section 3.8, the operator adjoint of Φ_o^+ is

$$\Phi_o^+(\underline{f})^* = \Phi_o^-(\underline{f}^*) \tag{219}$$

using the dual (8) for functions on the right-hand side and the Hilbert space operator adjoint on the left-hand side. The adjoint is

$$\widetilde{\Phi}_o^+(p)^* = \widetilde{\Phi}_o^-(-p)$$

from the Fourier transform (218) and $*$-dual (8). For $f \in \mathcal{P}(\mathbb{R}^4)$, $\Phi_o(\underline{f}) = \Phi_o^+(\underline{f})$ as a consequence of the support constraint on functions in $\underline{\mathcal{P}}$. From section 3.3.1, the commutators of

free field creation Φ_o^+ and annihilation Φ_o^- components provide that

$$[\Phi_o^-(x), \widetilde{\Phi_o^+}(p)] = \frac{\delta^+(p)e^{ipx}}{(2\pi)^2} \tag{220}$$

using the definition of Fourier transforms for generalized functions (218), the Fourier transform (16), and the notation

$$\delta^\pm(p) := \theta(\pm p_0)\delta(p^2 - \lambda_c^{-2})$$

from (37). The adjoint results in

$$[\widetilde{\Phi_o^-}(-p), \Phi_o^+(x)] = \frac{\delta^+(p)e^{-ipx}}{(2\pi)^2}.$$

The commutation relations and annihilation of the vacuum by Φ_o^- provide the convenient result

$$\langle \Omega_o | (\Phi_o^-(x))^k \prod_{j=1}^n \widetilde{\Phi_o^+}(p_j)\, \Omega_o \rangle = \frac{n!\, \delta_{k,n}}{(2\pi)^{2n}} \prod_{j=1}^n \delta^+(p_j) e^{ip_j x} \tag{221}$$

demonstrated by induction and using the Kronecker delta, $\delta_{k,n} = 0$.

The Feynman series for scattering amplitudes results from the Neumann series solution with the Dyson operator substituted into the scattering amplitudes (217). The Dyson operator is

$$\begin{aligned} U_D(\lambda_1, \lambda_2) &:= e^{iH_0\lambda_1} e^{-iH(\lambda_1 - \lambda_2)} e^{-iH_0\lambda_2} \\ &= \mathbb{I} - i\int_{\lambda_2}^{\lambda_1} ds\; H_{int}(s)\, U_D(s, \lambda_2) \end{aligned} \tag{222}$$

evaluated with $\lambda_1 = -\lambda_2 \to \infty$. The resulting Volterra equation of the second kind for $U_D(\lambda_1, \lambda_2)$ is satisfied formally by the Dyson operator. In (217), $\Phi_o(x)$ is a neutral scalar free quantum field and H_0 generates time translations of these free fields. The Hamiltonian H is expressed in free fields.

$$H := H_0 + H_{int}$$

with a conjectured interaction Hamiltonian for a self-interacting, neutral scalar field of

$$H_{int}(x_0) = \sum_{\ell \geq 4} a_\ell \int d\mathbf{x}\; :(\Phi_o(x_0, \mathbf{x}))^\ell : . \tag{223}$$

The summation is over all $\mathbf{x} \in \mathbb{R}^3$. The notation $:(\Phi_o)^\ell:$ designates *normal ordering* of the Hamiltonian (223). Normal ordering designates that the factors of Φ_o^+ and Φ_o^- in the binomial expansion of $(\Phi_o)^\ell$ are ordered with every Φ_o^- to the right of any Φ_o^+ [7, 23, 52, 61]. Normal ordering sets $\langle \Omega_o | H_{int}\, \Omega_o \rangle = 0$ but does not place the vacuum $|\Omega_o\rangle$ in the null space of the

Hamiltonian, $|H_{int}\,\Omega_o\rangle \neq 0$. From the Campbell-Baker-Hausdorff expression for this example of a boson field,

$$\begin{aligned} :\exp(\alpha\Phi_o): &= :\exp(\alpha\Phi_o^- + \alpha\Phi_o^+): \\ &= \exp(\alpha\Phi_o^-)\exp(\alpha\Phi_o^+) \\ &= \exp(\alpha\Phi_o^- + \alpha\Phi_o^+)\exp(-\tfrac{\alpha^2}{2}[\Phi_o^-, \Phi_o^+]). \end{aligned}$$

From section 3.3.1, the commutator is central, commutes with both Φ_o^+ and Φ_o^-, in the algebra generated by Φ_o^+ and Φ_o^-. Note that $[\Phi_o^-(x), \Phi_o^+(y)]$ diverges for $x = y$ and this is one of many divergences encountered in the elevation of classical to quantum Hamiltonian in the canonical formalism. Normal ordering is not a linear operation on the algebra generated from Φ_o^+ and Φ_o^-. A contradiction to a general specification for normal order as a linear operation in the algebra of fields is illustrated by the free field commutation relation

$$[\Phi_o^-(\underline{f}_1), \Phi_o(\underline{f}_2)] = W_2(f_1\, f_2).$$

Normal ordering produces

$$:[\Phi_o^-(\underline{f}_1), \Phi_o^+(\underline{f}_2)]: = 0 \neq :W_2(f_1\, f_2): = W_2(f_1\, f_2).$$

The choice of Hamiltonian (223) associates RQFT with a classical field model. Normal ordering determines an order for the non-commuting operators that elevate commuting classical dynamical variables.

The first contributing order of the Neumann series for the Dyson operator $U_D(\lambda_1, \lambda_2)$ is to approximate

$$U_D(\lambda_1, \lambda_2) \approx \mathbb{I} - i\int_{\lambda_2}^{\lambda_1} ds\; H_{int}(s).$$

Then, the first contributing order to the Feynman rules scattering amplitude (217) is the generalized function

$$\begin{aligned} S_{k,n-k}((p)_n) &= \langle \widetilde{\Phi_o^+}(p_k)\ldots\widetilde{\Phi_o^+}(p_1)\Omega_o | U_D(\infty,-\infty)\widetilde{\Phi_o^+}(p_{k+1})\ldots\widetilde{\Phi_o^+}(p_n)\Omega_o\rangle \\ &\approx \langle\Omega_o|\prod_{j=1}^{k}\widetilde{\Phi_o^-}(-p_j)\;(\mathbb{I} - i\int_{-\infty}^{\infty} ds\; H_{int}(s))\prod_{j=k+1}^{n}\widetilde{\Phi_o^+}(p_j)\Omega_o\rangle \\ &= \langle\Omega_o|\prod_{j=1}^{k}\widetilde{\Phi_o^-}(-p_j)\;(\mathbb{I} - i\sum_{\ell} a_\ell \int dx\sum_{\nu}\tbinom{\ell}{\nu}(\Phi_o^+(x))^{\nu}(\Phi_o^-(x))^{\ell-\nu})\prod_{j=k+1}^{n}\widetilde{\Phi_o^+}(p_j)\Omega_o\rangle. \end{aligned}$$

The normal ordered binomial expansion and commutation relations (220) provide that the only term that does not include forward scattering contributions has $\ell = n$ and $\nu = k$. Other

terms include forward contributions and are discarded in this characterization of the connected function ${}^{C}\mathcal{W}_{k,n-k}$ associated with a non-forward scattering amplitude. An associated connected VEV derives from the non-forward contribution to

$$\begin{aligned}
&-ia_n \tbinom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\, (\Phi_o^+(x))^k (\Phi_o^-(x))^{n-k} \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j) \Omega_o \rangle \\
&\quad = -ia_n \tbinom{n}{k} \int dx\ \langle \Omega_o | \prod_{j=1}^{k} \widetilde{\Phi_o^-}(-p_j)\, (\Phi_o^+(x))^k \Omega_o \rangle \langle \Omega_o (\Phi_o^-(x))^{n-k} \prod_{j=k+1}^{n} \widetilde{\Phi_o^+}(p_j) \Omega_o \rangle \\
&\quad = -i \frac{n!\, a_n}{(2\pi)^{2n-4}}\ \delta(-p_1 \ldots - p_k + p_{k+1} \ldots + p_n) \prod_{\ell=1}^{n} \delta^+(p_\ell) \\
&\quad = -ic_n\ \delta(-p_1 \ldots - p_k + p_{k+1} \ldots + p_n) \prod_{j=1}^{n} \delta(p_j^2 - \lambda_c^{-2}) \\
&\quad = -i\ {}^{C}\widetilde{\mathcal{W}}_{k,n-k}((-p)_k, (p)_{k+1,n}).
\end{aligned}$$

Commutations of $\widetilde{\Phi_o^-}(-p_j)$ with $\widetilde{\Phi_o^+}(p_\ell)$ produce forward contributions and are discarded in the second line. Substitution of (221) and the Fourier expansion of the delta function results in the third line. Equality of the third lines and fourth lines applies for the energy support constrained functions from the completion of basis spaces $\underline{\mathcal{P}}$, and not for all tempered functions $\underline{\mathcal{S}}$. Factors $\theta(E_j)$ in the $\delta^+(p_j)$ are redundant with the support constraints for either the appropriate function in $\underline{\mathcal{P}}$ or $*$-dual (8) of $\underline{\mathcal{P}}$. The coefficients a_ℓ in the Hamiltonian (223) and the c_n in the description of the VEV (53) are related

$$c_n := \frac{n!\, a_n}{(2\pi)^{2n-4}}.$$

In the final line, a neutral scalar field connected VEV ${}^{C}\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.4 is identified and substituted. For Hamiltonians (223), the first order contribution to the Feynman series scattering amplitude (217) coincides with the Fourier transform of a connected VEV ${}^{C}\widetilde{\mathcal{W}}_{k,n-k}$ from section 3.4. While the Hamiltonian (92) of the construction is distinct from the conjectured canonical formalism Hamiltonian (223), the weak coupling scattering amplitudes of the two developments nearly coincide: the Feynman series continues with renormalized, higher powers of the interaction Hamiltonian. The phase 'i' is irrelevant except for phase differences between forward and non-forward contributions, and the effects of these phases vanish in the scattering limit. The phase difference implements nonnegativity of the scalar product A.2.

The first contributing order of the scattering amplitudes (217) coincide up to a phase with the constructed scattering amplitudes (98) for a neutral scalar field. Then, the scattering amplitudes of low order expansion in RQFT and the constructions are the same: the Feynman

rules scattering amplitudes asymptotically coincide at weak coupling with the scattering amplitudes from the fully quantum mechanical constructions. This equality persists in non-scalar field examples if relativistic corrections to the Feynman series are made in appropriate cases [35].

A justification for the RQFT rule (217) from an asymptotic equality of free and interacting fields develops contradictions. If the free field were unitary similar to the interacting field at asymptotic times, $\lambda \to \pm\infty$,

$$e^{-iH\lambda}\Phi(\underline{f})e^{iH\lambda} = e^{-iH_0\lambda}\Phi_o(\underline{f})e^{iH_0\lambda}$$

with

$$\Omega_o = e^{iH\lambda}e^{-iH_0\lambda}\Omega,$$

then the scattering amplitude (217) would follow. The unitary similarity of interacting and free fields, and translation invariance of vacuums would provide that

$$\begin{aligned}\langle\Phi(\underline{f}_k)\dots\Phi(\underline{f}_1)\Omega|\Phi(\underline{f}_{k+1})\dots\Phi(\underline{f}_n)\Omega\rangle \\ = \langle\Phi_o(\underline{f}_k)\dots\Phi_o(\underline{f}_1)\Omega_o|U_D(\lambda_{out},\lambda_{in})\Phi_o(\underline{f}_{k+1})\dots\Phi_o(\underline{f}_n)\Omega_o\rangle.\end{aligned} \tag{224}$$

However, neither precedent is true.

1. If a unitary similarity implied (224), that similarity would contradict the Haag (Haag-Hall-Wightman-Greenberg) theorem [9, 56, 64]. Haag's theorem demonstrates that unitary similarity of free and interacting fields is not possible. Unitary similarity applies to fields Φ and Φ_o that are densely defined operators in a common Hilbert space [24].

2. In the Fock space of the free field, translation invariance of the free field VEV (44) provides that the free field vacuum Ω_o is translation invariant. But, this vacuum is not invariant to $e^{iH\lambda}$, $H_{int}\Omega_o \neq 0$. The all creation operator term contributes. For example, from the commutation relation (220) and VEV (221), the Fourier transform of delta functions, and the annihilation of the vacuum by Φ_o^-, it follows that

$$\begin{aligned}\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|H_{int}\Omega_o\rangle &= a_k\langle\widetilde{\Phi_o^+}(p_k)\dots\widetilde{\Phi_o^+}(p_1)\Omega_o|\int d\mathbf{x}\ (\Phi_o^+(x))^k\Omega_o\rangle \\ &= \frac{k!\,a_k}{(2\pi)^{2k-3}}\ \delta(\mathbf{p}_1+\dots+\mathbf{p}_k)\prod_{j=1}^{k}e^{i\omega_j x_0}\delta^-(p_j) \\ &\neq 0\end{aligned}$$

for the Hamiltonians (223).

Higher order contributions to the Feynman series for scattering amplitudes (217) diverge but terms are renormalized (regularized) to achieve convergent contributions in each order of the iteration (217). If a finite number of regularizations suffice to achieve convergent terms to all orders, then the interaction Hamiltonian is denoted *renormalizable*. Indications are that summations of the regularized Feynman series diverge [56].

6.9 The degenerate scalar product for free field VEV

It is well established [9, 56] that the sesquilinear form (21) defines a degenerate scalar product for function sequences from $\underline{\mathcal{S}}$ with free field VEV (38). In this appendix, the demonstration that the free field VEV define a degenerate scalar product for function sequences from $\underline{\mathcal{F}}$ is replicated in the notation and methods of these notes. The sesquilinearity of the form (21) is verified by inspection, and it suffices to demonstrate that the sesquilinear form is nonnegative.

From the result (68) that

$$\underline{\mathcal{F}} = \exp\circ({}^{C}\underline{\mathcal{F}})$$

with

$${}^{C}\underline{\mathcal{F}} = (0, 0, W_2, 0, \ldots),$$

in section 3.4.4, that positively weighted linear combinations of function sequences generate degenerate scalar products if the terms individually generate degenerate scalar products (21), that the $\circ$-product preserves signed symmetry, section 3.4.4, and that the $\circ$-product preserves nonnegativity of signed symmetric factor sequences, section 3.5.1, a demonstration of the nonnegativity for ${}^{C}\underline{\mathcal{F}}$ suffices to demonstrate the nonnegativity of $\underline{\mathcal{F}}$. The nonnegativity for ${}^{C}\underline{\mathcal{F}}$ follows immediately from the nonnegativity of the free field two-point function (37). The alternative demonstration below validates results of sections 3.4.4 and 3.5.1.

Every state labeled by a sequence from $\underline{\mathcal{S}}$ is a linear combination of states labeled by function sequences $\underline{\mathcal{P}}$. Discussed in [31] and applying the development of section 3.3.1, this result follows from the commutation relations

$$[\Phi^-(f_1)_{\kappa_1}, \Phi^+(f_2)_{\kappa_2}]_\pm = W_2(f_1\, f_2)_{\kappa_1\kappa_2},$$

the properties of the vacuum state, and the limitation of the energy support of the creation component to positive energies. There is a cyclic vacuum state with $\langle\Omega|\ldots\Phi^-(f)_\kappa\Omega\rangle = 0$ and $\langle\Omega|\Phi^+(f)_\kappa\ldots\Omega\rangle = 0$. The expansion of a state described by an $\underline{f} \in \underline{\mathcal{S}}$ into a description by an $\underline{g} \in \underline{\mathcal{P}}$ results from normal ordering the factors of the creation and annihilation constituents of the field, $\Phi(f)_\kappa = \Phi^-(f)_\kappa + \Phi^+(f)_\kappa$. As a consequence, a demonstration of the nonnegativity of the degenerate scalar product (21) for function sequences $\underline{\mathcal{P}}$ suffices for the free field VEV.

From (44) in section 3.3.1, the free field VEV for $\underline{\mathcal{P}}$ are

$$\begin{aligned}\mathcal{F}_{k,k}(\{1,2k\}) &= \sum_S \sigma(S) \prod_{j=1}^{k} W_2(\{j,i_j\}) \\ &= \frac{1}{k!}\sum_S \sum_{S'} \sigma(S)\sigma(S') \prod_{j=1}^{k} W_2(\{i'_j,i_j\})\end{aligned} \tag{225}$$

with the abbreviated notation $\sigma(S) := \sigma(S,(\kappa)_n)$ for signs from section 3.4. The summation $\sum_S$ includes the $k!$ distinct orderings of $\{k+1,2k\}$ and result in $k!$ distinct pairings j,i_j with $j \in \{1,k\}$ and $i_j \in \{k+1,2k\}$. The summation $\sum_{S'}$ includes the $k!$ distinct orderings of $\{1,k\}$. From the discussion of section 3.4, symmetry of $\mathcal{F}_{k,k}(\{1,2k\})$ under the indicated transpositions of arguments and that $\sigma(S,(\kappa)_n)^2 = 1$ results in equality of the two expressions. $S = \{1,\dots k,i_1,\dots i_k\}$, $S' = \{i'_1,\dots i'_k,k+1,\dots 2k\}$ and evaluation of the signs $\sigma(S,(\kappa)_{2k}) = \pm 1$ is discussed in (62) with $\sigma(S_o) = 1$ for $S_o = \{1,\dots k,k+1,\dots 2k\}$.

The nonnegativity of the matrix $DM(p)$ results in a factorization (39), $DM(p) = C^\dagger(p)C(p)$ with D the Dirac conjugation matrix from (8) and $C^\dagger(p)$ is the Hermitian transpose of $C(p)$. Substituted into (37), this factorization results in

$$\begin{aligned}DW_2(x_1,x_2)_{\kappa_1\kappa_2} &= \int dp_1 dp_2\; e^{ip_1x_1+ip_2x_2}\delta(p_1+p_2)\;\delta_2^+\, DM(p_2)_{\kappa_1\kappa_2} \\ &= \sum_{\ell=1}^{N_c}\int dp_2\; e^{-ip_2(x_1-x_2)}\;\delta_2^+\,\overline{C}(p_2)_{\ell\kappa_1}C(p_2))_{\ell\kappa_2}.\end{aligned} \tag{226}$$

With the designation of symmetrized functions

$$h_k((x)_k)_{(\kappa)_k} := \sum_S \sigma(S)_{(\kappa)_k} f_k(\{i_1,\dots i_k\})$$

and for the free field VEV (225), the sesquilinear form (21) becomes

$$\begin{aligned}\mathcal{F}(\underline{f}^*,\underline{f}) &= \sum_{n,m}\sum_{(\kappa)_{n+m}}\int d(x)_{n+m}\;(D\cdot)_n\mathcal{F}_{n,m}((x)_{n+m})_{(\kappa)_{n+m}} \\ &\qquad\qquad \times\overline{f_n}(x_n,\dots x_1)_{\kappa_n\dots\kappa_1}\, f_m(x_{n+1},\dots x_{n+m})_{(\kappa)_{n+1,n+m}} \\ &= \sum_k \frac{1}{k!}\sum_{(\kappa)_{2k}}\int d(x)_{2k}\prod_{j=1}^{k} DW_2(k+1-j,k+j) \\ &\qquad\qquad \times\overline{h_k}(x_k,\dots x_1)_{\kappa_k\dots\kappa_1}\, h_k(x_{k+1},\dots x_{2k})_{(\kappa)_{k+1,2k}}.\end{aligned}$$

Relabeling summation variables, the factorization (226) and the Fourier transform (16) produce

$$\begin{aligned}\mathcal{F}(\underline{f}^*,\underline{f}) &= \sum_k \frac{1}{k!} \sum_{(\kappa)_{2k}} \int d(x)_{2k} \prod_{j=1}^{k} \Big(\sum_{\ell_j=1}^{N_c} \int dp_j \; e^{ip_j(x_j - x_{k+j})} \delta_j^+ \overline{C}_{\ell_j\kappa_j}(p_j) C_{\ell_j\kappa_{k+j}}(p_j)\Big) \\ &\qquad \times \overline{h_k}(x_1,\ldots x_k)_{\kappa_1\ldots\kappa_k} \, h_k(x_{k+1},\ldots x_{2k})_{(\kappa)_{k+1,2k}} \\ &= \sum_k \frac{1}{k!} \sum_{(\ell)_k} \int d(p)_k \; \left(\prod_{i=1}^{k} \delta_i^+\right) \left| \sum_{(\kappa)_k} \left(\prod_{j=1}^{k} C_{\ell_j\kappa_j}(p_j)\right) \tilde{h}_k((p)_k)_{(\kappa)_k} \right|^2\end{aligned}$$

that is manifestly nonnegative and completes the demonstration.

6.10 Temporal evolution of function supports

For every $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$,

$$\tilde{\varphi}(p) = (p_0 + \omega) e^{-iE\lambda} \tilde{f}(\mathbf{p})$$

is a function within $\mathbf{H}_{\mathcal{P}}$. λ is real parameter that translates the temporal support. The inverse Fourier transform of the state describing function $\tilde{\varphi}$ is

$$\begin{aligned}\varphi(x) &= \int \frac{dp}{(2\pi)^2} \; e^{ipx} (p_0 + \omega) \tilde{f}(\mathbf{p}) \\ &= \left(-i\frac{\partial}{\partial x_0} + \sqrt{\lambda_c^{-2} - \Delta}\right) \int \frac{dp_0}{(2\pi)^{\frac{1}{2}}} \; e^{ip_0(x_0-\lambda)} \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \; e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda} \tilde{f}(\mathbf{p}) \qquad (227) \\ &= (2\pi)^{\frac{1}{2}} \left(-i\frac{\partial}{\partial x_0} + \sqrt{\lambda_c^{-2} - \Delta}\right) \delta(x_0 - \lambda) f(\lambda, \mathbf{x})\end{aligned}$$

with Δ the Laplacian for $\mathbf{R}^3$, the derivative of $\delta(t)$ is a generalized function [19], and the function $f(\lambda, \mathbf{x}) \in \mathcal{S}(\mathbb{R}^3)$ for any real λ.

$$f(\lambda, \mathbf{x}) := \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \; e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\omega\lambda} \tilde{f}(\mathbf{p}) \qquad (228)$$

with $\omega = \omega(\mathbf{p})$ from (13) and $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$. In the conventions of section 4.2, $\varphi(x)$ defines functions supported on time $x_0 = \lambda$ with spatial support described by $f(\lambda, \mathbf{x})$.

For a rotationally invariant function $\tilde{f}$, the z-axis of the parametrization of $\mathbf{p}$ can be aligned with the z-axis for any selected spatial vector $\mathbf{x}$. In this instance, changing summation variables to polar coordinates results in

$$\mathbf{p} \cdot \mathbf{x} = \rho r \cos\phi$$

with

$$\begin{aligned} \rho^2 &:= \mathbf{p}^2 \\ r^2 &:= \mathbf{x}^2 \end{aligned}$$

and

$$f(\lambda, \mathbf{x}) := \int_0^\infty \rho^2 d\rho \int_0^\pi \sin\phi d\phi \int_0^{2\pi} \frac{d\theta}{(2\pi)^{\frac{3}{2}}} \; e^{-i\rho r\cos\phi} e^{i\omega\lambda} \tilde{f}(\mathbf{p}).$$

The Cartesian components of these polar coordinates are

$$\mathbf{p} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}. \tag{229}$$

The angle summations are elementary leaving a single summation to evaluate (228) for a rotationally symmetric, even $\tilde{f}(\mathbf{p})$.

$$\begin{aligned} f(\lambda, \mathbf{x}) &= \frac{-i}{\sqrt{2\pi}\, r} \int_0^\infty \rho d\rho \; \left(e^{i\rho r} - e^{-i\rho r} \right) e^{i\omega\lambda} \tilde{f}(\rho) \\ &= \frac{-i}{\sqrt{2\pi}\, r} \int_{-\infty}^\infty \rho d\rho \; e^{i\rho r} e^{i\omega\lambda} \tilde{f}(\rho) \\ &= -\frac{1}{\sqrt{2\pi}\, r} \frac{\partial}{\partial r} \int_{-\infty}^\infty d\rho \; e^{i\rho r} e^{i\omega\lambda} \tilde{f}(\rho) \end{aligned} \tag{230}$$

from a change of summation variable $\rho \to -\rho$ in the second term and the dominated convergence theorem.

Nonrelativistically supported Gaussian functions provide an elementary example. Gaussian, minimum uncertainty functions centered of zero momentum,

$$\tilde{f}(\mathbf{p}) := e^{-\sigma^2 \mathbf{p}^2}, \tag{231}$$

are even and rotationally symmetric with a length parameter σ that characterizes the breadth of the spatial support of (228). Gaussian functions achieve the Heisenberg uncertainty lower bound on location and momentum support spread. If the support of $\tilde{f}(\mathbf{p})$ is over nonrelativistic ρ, $\mathbf{p}^2 = \rho^2 \ll \lambda_c^{-2}$ within the dominant support, then

$$\omega \approx \lambda_c^{-1} + \frac{1}{2}\lambda_c \rho^2 - \frac{1}{8}\lambda_c^3 \rho^4 \ldots$$

from Taylor series expansion of (13). If λ is sufficiently small, then

$$\exp(i\omega\lambda) \approx \exp(i\lambda_c^{-1}\lambda + \frac{i}{2}\lambda_c \rho^2 \lambda).$$

λ is sufficiently small if $\lambda_c^3\rho^4\lambda \ll 16\pi$ within the dominant support of $\tilde{f}(\mathbf{p})$, and this is implied if

$$\lambda \ll 16\pi\lambda_c$$

since $\sigma^2 \gg \lambda_c^2$ for nonrelativistically supported $\tilde{f}(\mathbf{p})$ and $\sigma\rho < \kappa$ for $\kappa \sim 10$ dependent on the confidence in likelihood. For nonrelativistically supported $\tilde{f}(\mathbf{p})$ and sufficiently small λ, (230) is approximated with $\omega \approx \lambda_c^{-1} + \frac{1}{2}\lambda_c\rho^2$,

$$\begin{aligned} f(\lambda, \mathbf{x}) &\approx -\frac{e^{i\frac{\lambda}{\lambda_c}}}{\sqrt{2\pi}\, r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\rho\; e^{i\rho r} e^{-(\sigma^2 - i\frac{1}{2}\lambda_c\lambda)\rho^2} \\ &= -\frac{e^{i\frac{\lambda}{\lambda_c}}}{\sqrt{2}\, r\,(\sigma^2 - i\frac{1}{2}\lambda_c\lambda)^{\frac{1}{2}}}\frac{\partial}{\partial r}\; e^{-r^2/(4(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))} \\ &= \frac{e^{i\frac{\lambda}{\lambda_c}}}{(2(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))^{\frac{3}{2}}}\; e^{-r^2/(4(\sigma^2 - i\frac{1}{2}\lambda_c\lambda))} \end{aligned}$$

from the Gaussian summation (170).

A family of even, rotationally symmetric $\tilde{f}(\mathbf{p})$ with $\tilde{\varphi}_2 \in \mathbf{H}_{\mathcal{P}}$ and relativistic time translations that remain in family are

$$\tilde{f}(\mathbf{p}) = \tilde{h}(\rho^2) e^{-2\sigma^2\lambda_c^{-2}(\lambda_c\omega - 1)}$$

for continuously differentable, polynomially bounded growth functions $\tilde{h}(\rho^2)$. The time translates have complex spread parameters σ^2 and a phase shift independent of $\mathbf{p}$.

$$e^{-i\omega\lambda}\tilde{f}(\mathbf{p}) = \tilde{h}(\rho^2) e^{-(2\sigma^2\lambda_c^{-2} + i\lambda_c^{-1}\lambda)(\lambda_c\omega - 1)} e^{-i\lambda_c^{-1}\lambda}.$$

If $\tilde{h}(\rho^2) \approx 1$ for $\lambda_c\rho \ll 1$, these $\tilde{f}(\mathbf{p})$ approximate the Gaussian functions (231)

$$\tilde{f}(\mathbf{p}) \approx e^{-\sigma^2\rho^2}$$

if momentum support is nonrelativistic, $\lambda_c\rho \ll 1$ within the dominant support of $\tilde{f}(\mathbf{p})$. The approximation derives from (13),

$$\lambda_c\omega \approx 1 + \frac{(\lambda_c\rho)^2}{2}.$$

In very relatistic instances $\omega \approx \rho$ and $f(\lambda, \mathbf{x})$ becomes a more Lorentzian than Gaussian function over $r = \|\mathbf{x}\|$. Without approximation of the Hamiltonian, estimates for $f(\lambda, \mathbf{x})$ (230) apply for all λ.

6.11 Center-of-momentum reference frames

For any energy-momentum vector q with $q^2 > 0$ and momentum $\mathbf{q}$, Poincaré invariance of the scalar product can be exploited to transform to a primed inertial reference frame with $\mathbf{q}$ transformed to $\mathbf{q}' = 0$. In particular, Poincaré invariance of the scalar product can be exploited to transform to the center-of-momentum frame for n particles, a reference frame with $\mathbf{p}_1 + \mathbf{p}_2 + \ldots \mathbf{p}_n = 0$. Two particles of energy-momenta q_1, q_2 with $q_1^2 = q_2^2 = m^2$ and momenta $\mathbf{q}_1, \mathbf{q}_2$ illustrate the result. There is a Lorentz boost to a reference frame with $\mathbf{q}_2' = -\mathbf{q}_1'$. The Poincaré transformation to a center-of-momentum frame is developed in this appendix. A zero momentum is unaffected by a coordinate frame translation. Following a boost with translation, the center-of-mass of n particles may be colocated with the origin of coordinates in a center-of-momentum frame.

The transformation from the original energy-momentum coordinates to coordinates in the primed reference frame is

$$p_j' = \Lambda p_j$$

with Λ a Lorentz transform. The center of momentum frame is defined by this linear transformation Λ that sets

$$\Lambda q_1 = (\omega(\mathbf{q}_1'), \mathbf{q}_1') \quad \text{and} \qquad \Lambda q_2 = (\omega(\mathbf{q}_1'), -\mathbf{q}_1').$$

Both q_1 and q_2 are on mass shells defined by finite rest mass m. The transformation Λ is a proper ($\det(\Lambda)$=1), orthochronous ($\Lambda_{00} > 0$) Lorentz transformation

$$\Lambda := \mathcal{B}(\beta)\mathcal{R}$$

consisting of a rotation $\mathcal{R}$ and a boost $\mathcal{B}(\beta)$. To evaluate Λ, designate the center-of-momentum for q_1, q_2 by

$$q := q_1 + q_2 = \begin{pmatrix} \omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) \\ \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix} \tag{232}$$

in polar coordinates with the momentum $\mathbf{q} := (\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z)$,

$$\rho := \|\mathbf{q}\| = \|\mathbf{q}_1 + \mathbf{q}_2\|,$$

and

$$\begin{aligned} \cos\phi &= \frac{\mathbf{q}_z}{\rho}, & \sin\phi &= \frac{\sqrt{\mathbf{q}_x^2 + \mathbf{q}_y^2}}{\rho} \\ \cos\theta &= \frac{\mathbf{q}_x}{\sqrt{\mathbf{q}_x^2 + \mathbf{q}_y^2}}, & \sin\theta &= \frac{\mathbf{q}_y}{\sqrt{\mathbf{q}_x^2 + \mathbf{q}_y^2}} \end{aligned} \tag{233}$$

with quadrants selected for θ, ϕ to correspond with the signs of $\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_z$. $\theta \in \{0, 2\pi\}$ is the anticlockwise angle of $\mathbf{q}$ from the x-axis in the x-y plane, and $\phi \in \{0, \pi\}$ is the angle of $\mathbf{q}$ from the z-axis in the plane containing $\mathbf{q}$ and the z-axis. The rotation $\mathcal{R}$ aligns the momentum $\mathbf{q}$ with the primed z-axis,

$$\begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ 0 \\ 0 \\ \rho \end{pmatrix} = \mathcal{R}q := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ 0 & \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}$$

and the boost $\mathcal{B}(\beta)$ zeros the momentum.

$$\begin{pmatrix} \omega(\mathbf{q}_1') + \omega(\mathbf{q}_2') \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(\beta) & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & \gamma(\beta) \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1)+\omega(\mathbf{q}_2) \\ 0 \\ 0 \\ \rho \end{pmatrix}$$

with

$$\beta := \frac{\rho}{\sqrt{(\omega(\mathbf{q}_1) + \omega(\mathbf{q}_2))^2 - \rho^2}}$$

and $\gamma(\beta) = \sqrt{1+\beta^2}$.

The transformation to the center-of-momentum frame determined by $\mathbf{q}_1$ and $\mathbf{q}_2$ is the Lorentz transform $\Lambda := \mathcal{B}(\beta)\mathcal{R}$.

$$\begin{pmatrix} \omega(\mathbf{q}_1') + \omega(\mathbf{q}_2') \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(\beta) & -\beta\frac{\mathbf{q}_x}{\rho} & -\beta\frac{\mathbf{q}_y}{\rho} & -\beta\frac{\mathbf{q}_z}{\rho} \\ 0 & \frac{\mathbf{q}_y}{\rho_{xy}} & -\frac{\mathbf{q}_x}{\rho_{xy}} & 0 \\ 0 & \frac{\mathbf{q}_x\mathbf{q}_z}{\rho_{xy}\rho} & \frac{\mathbf{q}_y\mathbf{q}_z}{\rho_{xy}\rho} & -\frac{\rho_{xy}}{\rho} \\ -\beta & \gamma(\beta)\frac{\mathbf{q}_x}{\rho} & \gamma(\beta)\frac{\mathbf{q}_y}{\rho} & \gamma(\beta)\frac{\mathbf{q}_z}{\rho} \end{pmatrix} \begin{pmatrix} \omega(\mathbf{q}_1) + \omega(\mathbf{q}_2) \\ \mathbf{q}_x \\ \mathbf{q}_y \\ \mathbf{q}_z \end{pmatrix} \tag{234}$$

with $\rho_{xy} := \sqrt{\mathbf{q}_x^2 + \mathbf{q}_y^2}$.

The inverse transformation Λ^{-1} is

$$\Lambda^{-1} = (B(\beta)R)^{-1} = R^T B(-\beta) = R^T B(-\beta)^T = (B(-\beta)R)^T$$

or

$$\Lambda^{-1} = \begin{pmatrix} \gamma(\beta) & 0 & 0 & \beta \\ \beta\frac{\mathbf{q}_x}{\rho} & \frac{\mathbf{q}_y}{\rho_{xy}} & \frac{\mathbf{q}_x\mathbf{q}_z}{\rho_{xy}\rho} & \gamma(\beta)\frac{\mathbf{q}_x}{\rho} \\ \beta\frac{\mathbf{q}_y}{\rho} & -\frac{\mathbf{q}_x}{\rho_{xy}} & \frac{\mathbf{q}_y\mathbf{q}_z}{\rho_{xy}\rho} & \gamma(\beta)\frac{\mathbf{q}_y}{\rho} \\ \beta\frac{\mathbf{q}_z}{\rho} & 0 & -\frac{\rho_{xy}}{\rho} & \gamma(\beta)\frac{\mathbf{q}_z}{\rho} \end{pmatrix}.$$

6.12 Two body classical trajectories

In this appendix, notation is established and results for two body trajectories from classical mechanics are collected.

Nonrelativistic two body problems provide explicit examples of classical trajectories $\mathbf{u}_j(\lambda)$. The motion of two classical bodies interacting by a scalar pair potential is executed within a plane. A pair potential depends solely on the body separation and constant properties of the bodies. The location of bodies in a plane at each time λ is specified by Cartesian spatial coordinates,

$$\mathbf{u}_j := \begin{pmatrix} r_j\cos\theta_j \\ r_j\sin\theta_j \\ 0 \end{pmatrix} \tag{235}$$

with $r_j := \|\mathbf{u}_j\|$ the Euclidean length. Notation is abbreviated, $\mathbf{u}_j = \mathbf{u}_j(\lambda)$ and similarly for r_j, θ_j.

The Lagrangian for a two body problem in the notation of this note is

$$\mathcal{L} = \frac{1}{2}m_1c^2\dot{\mathbf{u}}_1^2 + \frac{1}{2}m_2c^2\dot{\mathbf{u}}_2^2 - V(\|\mathbf{u}_1 - \mathbf{u}_2\|)$$

with V the pair potential. The separation of the two bodies is

$$\mathbf{u}_s = \mathbf{u}_1 - \mathbf{u}_2.$$

In a reference frame with the center-of-mass concident with the origin of coordinates,

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = 0.$$

The solutions for $\mathbf{u}_i$ given the separation $\mathbf{u}_s$ in the center-of-mass coordinate frame are

$$\mathbf{u}_1 = \frac{m_2\mathbf{u}_s}{m_1+m_2}, \qquad \mathbf{u}_2 = -\frac{m_1\mathbf{u}_s}{m_1+m_2}.$$

Substitution results in a single body Lagrangian for $\mathbf{u}_s$,

$$\mathcal{L} = \frac{1}{2}\mu c^2\dot{\mathbf{u}}_s^2 - V(\|\mathbf{u}_s\|),$$

with μ denoted the reduced mass.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This results in Newton's equation of motion for the separation,

$$\mu c^2 \ddot{\mathbf{u}}_s = -\frac{\partial V}{\partial \mathbf{u}_s} \tag{236}$$

with

$$\frac{\partial V}{\partial \mathbf{u}_s} = \nabla_{\mathbf{u}_s} V$$

a gradient vector. For equal masses in the center-of-mass reference frame,

$$\mu = \frac{m}{2}, \qquad \mathbf{u}_2 = -\mathbf{u}_1$$

and with a $-g/r$ pair potential, $V = -gmc^2/\|\mathbf{u}_s\|$, substitution of $2\|\mathbf{u}_1\| = \|\mathbf{u}_s\|$ for the separation results in equation of motion

$$\ddot{\mathbf{u}}_1 = -\frac{g}{4}\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|^3}. \tag{237}$$

The interaction is characterized by the length g. For gravity, $g = Gm^2/mc^2$ and for electrostatics $g = Ke^2/mc^2$. For a mass of one a.m.u. and an elementary charge of one, $g = 1.23 \times 10^{-53}$ m for gravity and $g = 1.54 \times 10^{-18}$ m for electrostatics.

With

$$L := \mathbf{u}_1^2 \dot{\theta}_1,$$

the relationships of Cartesian and radial coordinates for two equal mass bodies $m = m_1 = m_2$ in the center-of-mass reference frame ($\mathbf{u}_2 = -\mathbf{u}_1$) include

$$\begin{aligned} \mathbf{u}_j^2 &= r_j^2 \\ \dot{\mathbf{u}}_j \cdot \mathbf{u}_j &= \dot{r}_j r_j \\ \dot{\mathbf{u}}_j^2 &= \dot{r}_j^2 + \frac{L^2}{r_j^2}. \end{aligned} \tag{238}$$

The trajectory $\mathbf{u}_1$ in the center-of-mass reference frame for the pair of equal mass bodies interacting via a $-g/r$ pair potential are conveniently parametrized by $\theta := \theta_1$ from (235). A solution to Kepler's problem is

$$r_1 = \frac{L^2/4g}{1 - \epsilon_r \cos\theta}$$

and

$$\lambda(\theta) = \frac{1}{L}\int_{\theta_0}^{\theta} d\phi \; r_1(\phi)^2$$

with $\lambda(\theta_0) = 0$ and

$$\epsilon_r := \sqrt{1 + 32e_1L^2/g^2}.$$

For $e_1 > 0$, the solution is unbound and diverges when $\epsilon_r \cos\theta = 1$. For these unbound trajectories, θ is constrained to the interval $(\beta, 2\pi - \beta)$ with $\beta = \cos^{-1}(\epsilon_r^{-1})$. Bound states have $e_1 \leq 0$ and

$$e_1 \geq -\frac{g^2}{32L^2}.$$

$e_1 = -g^2/32L^2$ are the circular orbits. For the circular orbits of $-g/r$ pair potentials,

$$4L^2 = gr_1.$$

6.13 Nonrelativistic energy approximations

In this appendix, nonrelativistic approximations to the Hamiltonian (92) are developed.

A nonrelativistic approximation of the energies ω_k from (13) derives from the Taylor theorem polynomial approximation for momenta $\mathbf{p}_k$ near $\mathbf{q}_k$. To second order in $\mathbf{p}_k - \mathbf{q}_k$,

$$\omega_k \approx \omega(\mathbf{q}_k) + \frac{\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)}{\omega(\mathbf{q}_k)} + \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)} - \frac{(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2}{2\omega(\mathbf{q}_k)^3}$$

and the Cauchy-Schwarz-Bunyakovski inequality provides that

$$(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2 \leq \mathbf{q}_k^2 \, (\mathbf{p}_k - \mathbf{q}_k)^2.$$

For nonrelativistic velocities

$$\mathbf{q}_k^2 \ll \omega(\mathbf{q}_k)^2$$

and then

$$\frac{(\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k))^2}{2\omega(\mathbf{q}_k)^3} \ll \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)}.$$

This justifies neglect of the last term from the Taylor series in (239) if the support of state describing functions excludes relativistic momenta, if, for example, (144) of section 4.4 applies. The resulting nonrelativistic approximation for ω_k is

$$\omega_k \approx \omega(\mathbf{q}_k) + \frac{\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)}{\omega(\mathbf{q}_k)} + \frac{(\mathbf{p}_k - \mathbf{q}_k)^2}{2\omega(\mathbf{q}_k)}. \tag{239}$$

The approximation applies for nonrelativistically supported state describing functions. If $\mathbf{q}_k$ is a representative for the support of a state describing function and $\mathbf{p}$ is any point from the nonrelativistic dominant support, then $\|\mathbf{p}_k - \mathbf{q}_k\| \ll \lambda_c^{-1} \leq \omega(\mathbf{q}_k)$. (239) provides a convenient approximation for the Hamiltonian (92) applied in each n-argument subspace,

$$H = \sum_k \omega_k.$$

If $\mathbf{q}_k$ and $\mathbf{p}_k$ are both nonrelativistic, $\|\mathbf{q}_k\|, \|\mathbf{p}_k\| \ll \lambda_c^{-1}$, then both correction terms to $\omega(\mathbf{q}_k)$ in (239) are second order in small quantities. Neglect of the term proportional to $(\mathbf{p}_2 - \mathbf{q}_k)^2$ in (142) results in a convenient linear in $\mathbf{p}_2$ approximation for the energy. However, with nonrelativistic momenta, this approximation is also second order in small quantities and neglect of the quadratic term is not justified. Neglect of the second correction requires that $(\mathbf{p}_k - \mathbf{q}_k)^2 \ll |\mathbf{q}_k \cdot (\mathbf{p}_k - \mathbf{q}_k)|$ in addition to nonrelativistic momenta.

A nonrelativistic approximation (109) of the energies $\omega(\mathbf{q}_k)$ also derives from a Taylor theorem polynomial approximation if $\|\mathbf{q}_k\| \ll \omega(\mathbf{q}_k)$.

$$\begin{aligned} \omega(\mathbf{q}_k) &\approx \frac{1}{\lambda_c} + \frac{\lambda_c \mathbf{q}_k^2}{2} \\ &= \frac{1}{\lambda_c}\left(1 + \frac{\dot{\mathbf{u}}_k(\lambda)^2}{2}\right) \end{aligned}$$

from the nonrelativistic relation between momentum and velocity (109) and with the reduced Compton wavelength (14). This approximation applies for nonrelativistic velocities of the corresponding classical trajectories $\mathbf{u}_k(\lambda)$ in appropriate reference frames.

The center-of-mass and the internal motion of bodies decouple in nonrelativistic instances. For two argument functions (134) in section 4.4, the assumption that decouples the motions is that the momentum of the center-of-momentum $\mathbf{p}_3' = \mathbf{p}_3 + \mathbf{p}_4$ from (134) in section 4.4 is nonrelativistic. Then, from (109),

$$\mathbf{p}_3'^2 \ll \frac{1}{\lambda_c^2} < \omega(\mathbf{p}_3')^2.$$

In these nonrelativistic instances, Taylor expansion results in

$$\begin{aligned} \omega_3 + \omega_4 &= \omega(\tfrac{\mathbf{p}_3' + \mathbf{p}_4'}{2}) + \omega(\tfrac{\mathbf{p}_3' - \mathbf{p}_4'}{2}) \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}_4') + \frac{\mathbf{p}_3'^2}{4\omega(\frac{1}{2}\mathbf{p}_4')} \\ &\approx 2\omega(\tfrac{1}{2}\mathbf{p}_4') + \tfrac{1}{4}\lambda_c \mathbf{p}_3'^2 \end{aligned}$$

using (13), (109) and (239). In this nonrelativistic approximation, the argument of the conservation of energy delta function becomes independent of $\mathbf{p}_1'$ and $\mathbf{p}_3'$ as a consequence of momentum conservation, $\mathbf{p}_1' = \mathbf{p}_3'$.

$$\begin{aligned} \omega_1 + \omega_2 - \omega_3 - \omega_4 &\approx 2\omega(\tfrac{1}{2}\mathbf{p}_2') + \tfrac{1}{4}\lambda_c \mathbf{p}_1'^2 - 2\omega(\tfrac{1}{2}\mathbf{p}_4') - \tfrac{1}{4}\lambda_c \mathbf{p}_3'^2 \\ &= 2\omega(\tfrac{1}{2}\mathbf{p}_2') - 2\omega(\tfrac{1}{2}\mathbf{p}_4'). \end{aligned} \tag{240}$$

This approximation appears in the energy conservation delta function independently of the propagation interval λ, unlike the generation of time translation with λ multiplying any error in the Hamiltonian.

6.14 Essentially local functions

The physical understanding of state descriptions in these notes uses that there are functions arbitrarily dominantly supported within finite volumes among the anti-local functions $\mathcal{P}(\mathbb{R}^4)$. These functions are essentially localized. There are no strictly localized functions in $\mathcal{P}(\mathbb{R}^4)$, [33].

H. Reeh and S. Schlieder [45, 53] demonstrated that the operation $(a^2 - \Delta)^{\frac{1}{2}}$ over $\mathbb{R}^3$ has the *anti-local property*: if both f and $(a^2 - \Delta)^{\frac{1}{2}} f$ vanish within some finite volume of $\mathbb{R}^3$, then $f \in \mathcal{L}^2(\mathbb{R}^3)$ is identically zero. $\varphi \in \mathcal{P}(\mathbb{R}^4)$ has a Fourier transform of the form

$$\tilde{\varphi}(p) = (p_0 + \omega)\, \tilde{g}(p)$$

with $g \in \mathcal{S}(\mathbb{R}^4)$ and $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$ from (13). Then

$$\varphi(x) = -i \frac{dg(x)}{dx_0} + (\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g(x)$$

and φ vanishing in a finite volume provides that both $\dot{g}$ and $(\lambda_c^{-2} - \Delta)^{\frac{1}{2}} g$ vanish in the volume [33]. Anti-locality can be motivated by the observation that if $\varphi(x)$ was supported solely in a finite volume of $\mathbf{x} \in \mathbb{R}^3$, then the Fourier transform would be an entire analytic function of $\mathbf{p} \in \mathbb{R}^3$, but both $p_0\, \tilde{g}(p)$ and $\omega\, \tilde{g}(p)$ can not both be entire due to the cut line of $\omega = (\lambda_c^{-2} + \mathbf{p}^2)^{\frac{1}{2}}$.

One example suffices to demonstrate the existence of essentially localized functions within $\mathcal{P}(\mathbb{R}^4)$. The selected example φ has a Fourier transform

$$\tilde{\varphi}(p) = (p_0 + \omega)\tilde{g}(p_0) \exp(-\alpha \mathbf{p}^2)$$

with $g \in \mathcal{S}(\mathbb{R})$. Then

$$\varphi(x) = \int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \left(-ig'(x_0) + \omega g(x_0)\right) \exp(-\alpha \mathbf{p}^2 - i\mathbf{p} \cdot \mathbf{x}) \tag{241}$$

from the properties (18) of the Fourier transform (16).

From the Gaussian summation (170) from section 4.4.4,

$$\begin{aligned}
\int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \exp(-\alpha\mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x}) &= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\phi d\phi \, \exp(-\alpha\rho^2 - i\rho r\sin\phi) \\
&= \int_0^\infty \frac{\rho^2 d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2) \frac{\left(e^{-i\rho r} - e^{i\rho r}\right)}{-i\rho r} \\
&= \frac{i}{r}\int_{-\infty}^\infty \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr}\int_{-\infty}^\infty \frac{d\rho}{(2\pi)^{\frac{1}{2}}} \exp(-\alpha\rho^2 - i\rho r) \\
&= -\frac{1}{r}\frac{d}{dr}\frac{1}{\sqrt{2\alpha}} \exp\left(-\frac{r^2}{4\alpha}\right) \\
&= (2\alpha)^{-\frac{3}{2}} \exp\left(-\frac{r^2}{4\alpha}\right)
\end{aligned}$$

using the spherical symmetry, change to polar coordinates with z-axis aligned with $\mathbf{x} = (0, 0, r)$,

$$\mathbf{p} = (\rho\cos\theta\cos\phi, \rho\sin\theta\cos\phi, \rho\sin\phi)$$

and ρ, r are the positive roots of the Euclidean lengths, $\rho^2 = \mathbf{p}^2$ and $r^2 = \mathbf{x}^2$, respectively. The change of variable $\rho' := -\rho$ in the second term in the third line and simplification results in the fourth line. In this case, the summation is the product of three one dimensional Gaussian summations (170) from section 4.4.4 but the development in polar coordinates is preparation for a second required summation. Similarly,

$$\begin{aligned}
\left|\int \frac{d\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \omega \exp(-\alpha\mathbf{p}^2 - i\mathbf{p}\cdot\mathbf{x})\right| &= \left|\frac{i}{r}\int_{-\infty}^\infty \frac{\rho d\rho}{(2\pi)^{\frac{1}{2}}} \omega \exp(-\alpha\rho^2 - i\rho r)\right| \\
&< \frac{1}{r}\int_{-\infty}^\infty \frac{|\rho| d\rho}{(2\pi)^{\frac{1}{2}}} \omega \exp(-\alpha\rho^2) \\
&= \frac{a}{r}
\end{aligned}$$

with the indicated constant a finite and independent of r. While apparently a loose upper bound, this bound suffices to demonstrate that the dominant support of $\varphi(x)$ from (241) lies within finite spheres: the likelihood per unit volume decreases as $1/r$ with expanding distance

r from the center of support. $\varphi(x)$ is finite for $r = 0$ and

$$\varphi(x) < \frac{|g'(x_0)|}{(2\alpha)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4\alpha}\right) + |g(x_0)| \, \frac{a}{r}$$

for every $g \in \mathcal{S}(\mathbb{R})$. Translations, dilations and any spherically symmetric $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ with essentially localized support substituted for the Gaussian function also have dominant support within a finite volume.

6.15 Example relative states

An example measurement process is illustrated in this appendix. The example illustrates properties of projection operators, unitary time translations, and relative states discussed in [11] and appendix 6.2.7. The example is a simplification generally consistent with the constructions but not derived from the state representations constructed in section 3.

Consider an observer with three orthogonal states of interest interacting with an observed system characterized by two orthogonal states. The three states of the observer are designated "no observation," "observed system state 1" and "observed system state 2." Designate the two system states as "up" and "down" regardless of whether they are states characterized by a spin. The two states could describe any system quantity characterized by two possibilities: located within detector A or B; Schrödinger's live or dead cat, spin up or down, and so forth. Then, there are six subspaces of states of interest within the Hilbert space if the observer is described independently of the system, for example, when the observer is distantly space-like separated from the system and not entangled. Designate these states using "no" for "no observation," "sup" for "observed system state 1" and "sdn" for "observed system state 2" with "up" and "dn" for the two system states. For an ideal measurement, $|\text{sdn}, \text{up}\rangle$ and $|\text{sup}, \text{dn}\rangle$ never appear in nature and these states are not coupled to the states of interest. These two states need not be considered further. This leaves four states of interest.

$$|\text{no}, \text{up}\rangle := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\text{no}, \text{dn}\rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\text{sup}, \text{up}\rangle := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\text{sdn}, \text{dn}\rangle := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (242)$$

Presumably, these states represent orthogonal subspaces of states $|\underline{f}\rangle$ labeled by particular function sequences $\underline{f}$, and their interaction is described by one of the VEV constructed in section 3.4. The observation is conceived as a scattering event: initially independently described system and observer are spatially distant, they approach and strongly interact, and then entangled system with observer states propagate away and cease to interact. An effective Hamiltonian is described below. The Hamiltonian is selected for simplicity with properties similar to the $-g/r$ potentials that describe long range, non-relativistic interactions in the constructions.

The evolution of an initial state described by the four orthogonal states (242) is described

$$|v(\lambda)\rangle := U(\lambda)|v_o\rangle$$

with a unitary, 4×4 block diagonal

$$U(\lambda) = \begin{pmatrix} U_2(\lambda) & 0 \\ 0 & U_2(\lambda) \end{pmatrix}$$

using 2×2 unitary

$$U_2(\lambda) = \begin{pmatrix} e^{i\eta}\cos\theta & -e^{i\phi-i\rho}\sin\theta \\ e^{i\rho}\sin\theta & e^{i\phi-i\eta}\cos\theta \end{pmatrix}.$$

The four parameters, θ, ϕ, η, ρ, are real. To simplify this example, the two unitary block submatrices are set equal with $\eta = \rho = \phi = 0$ and

$$\theta(\lambda) := \frac{\pi}{2}\frac{1}{I_\infty}\int_{-\infty}^{\lambda} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}}.$$

The normalization I_∞ is a beta function valid if $\epsilon > 0$.

$$I_\infty := \int_{-\infty}^{\infty} ds\ \frac{a^{2\epsilon}}{(a^2+s^2)^{\frac{1}{2}+\epsilon}} = \frac{\Gamma(\frac{1}{2})\Gamma(\epsilon)}{\Gamma(\frac{1}{2}+\epsilon)}.$$

$\theta(\lambda)$ is absolutely continuous with $\theta(-\infty) = 0$ and $\theta(\infty) = \pi/2$.

The Hamiltonian generates time translation.

$$\begin{aligned} H(\lambda)v(\lambda) &:= i\frac{dv(\lambda)}{d\lambda} \\ &= i\dot{U}(\lambda)v_o \\ &= i\dot{U}(\lambda)U^{-1}(\lambda)v(\lambda) \\ &= i\dot{U}(\lambda)U^{\dagger}(\lambda)v(\lambda) \end{aligned}$$

in units of inverse length. For the 2×2 blocks of the Hamiltonian,

$$\begin{aligned} H_2(\lambda) &= i\dot{U}_2(\lambda)U_2^{\dagger}(\lambda) \\ &= \begin{pmatrix} -i\dot{\theta}\sin\theta & -i\dot{\theta}\cos\theta \\ i\dot{\theta}\cos\theta & -i\dot{\theta}\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i\dot{\theta}(\lambda) \\ i\dot{\theta}(\lambda) & 0 \end{pmatrix} \end{aligned}$$

with

$$\dot{\theta}(\lambda) = \frac{\pi}{2}\,\frac{1}{I_\infty}\,\frac{a^{2\epsilon}}{(a^2+\lambda^2)^{\frac{1}{2}+\epsilon}}.$$

The Hamiltonian is Hermitian as a consequence of the unitarity of $U_2(\lambda)$. The evolution of the initial state v_o occurs most rapidly near $\lambda = 0$. The parameter a is a length characterizing the effective closest approach of observed to observer.

An initial state

$$|v_o\rangle = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ and normalization $|\alpha|^2 + |\beta|^2 = 1$ evolves to

$$|v(\lambda)\rangle = U(\lambda)|v_o\rangle = \begin{pmatrix} \alpha\cos\theta(\lambda) \\ \alpha\sin\theta(\lambda) \\ \beta\cos\theta(\lambda) \\ \beta\sin\theta(\lambda) \end{pmatrix} \tag{243}$$

with $\cos\theta(-\infty) = 1$, $\sin\theta(-\infty) = 0$ and $\cos\theta(\infty) = 0$, $\sin\theta(\infty) = 1$. Initially, no observation has occured and as time evolves, the system and observer approach and the likelihood of state transitions increases. Eventually, the observer and system separate and likelihoods stabilize with the relevant observer states entangled with system states.

The selected form for $U(\lambda)$ couples the observer with the system and lacks any evolution of "up" and "dn" system states, e.g., evolution of live into dead cats. A more general unitary transformation implements changes to the likelihoods of live or dead with time, for example, the composition of a rotation of $v_o = (1,0,0,0)$ to $(\cos\vartheta, 0, \sin\vartheta, 0)$ with $U(\lambda)$. If the transition is internal to the system, more general effective interactions apply than if the transition is due to interaction with the observer. In the case with the transition due to interaction with the observer, $\vartheta(\lambda)$ would have similar properties to $\theta(\lambda)$ except the transition need not be from one certain extreme to the other. $\vartheta(\lambda)$ is constrained by physical considerations, e.g., dead cats do not become live again. For this example composition, the Hamiltonian is time-dependent,

$$H(\lambda) = \begin{pmatrix} 0 & -i\dot{\theta} & -i\dot{\vartheta}\cos^2\theta & -i\dot{\vartheta}\cos\theta\sin\theta \\ i\dot{\theta} & 0 & -i\dot{\vartheta}\cos\theta\sin\theta & -i\dot{\vartheta}\sin^2\theta \\ i\dot{\vartheta}\cos^2\theta & i\dot{\vartheta}\cos\theta\sin\theta & 0 & -i\dot{\theta} \\ i\dot{\vartheta}\cos\theta\sin\theta & i\dot{\vartheta}\sin^2\theta & i\dot{\theta} & 0 \end{pmatrix}$$

and reproduces the result (243) but with $\alpha = \cos\vartheta(\lambda)$ and $\beta = \sin\vartheta(\lambda)$.

The projections onto the perceived system states are

$$P_{\rm up} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{\rm dn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the projections onto the possible states of the observer are

$$E_{\rm no} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\rm sup} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_{\rm sdn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These projections commute, $[E_{\rm xx}, P_{\rm yy}] = 0$. This commutation implies the decomposition into relative states discussed in [11] and appendix 6.2.7. The projections onto perceived system states, $P_{\rm yy}$, commute with time translation, $[U(\lambda), P_{\rm yy}] = 0$, providing that the likelihoods of the two system states do not vary with time. The projections onto the possibilities for observer state, $E_{\rm xx}$, do not commute with time translation and state of the observer evolves with time. Likelihoods are

$$\text{up likelihood } = E[P_{\rm up}] = \langle v(\lambda)|P_{\rm up}v(\lambda)\rangle = |\alpha|^2$$

and

$$\text{dn likelihood } = E[P_{\rm dn}] = \langle v(\lambda)|P_{\rm dn}v(\lambda)\rangle = |\beta|^2$$

both independently of λ in the first example. The likelihoods of the observer having the indicated perceived history of observations are

$$\begin{aligned} &\text{likelihood of no observation} = E[E_{\rm no}] = \langle v(\lambda)|E_{\rm no}v(\lambda)\rangle = \cos^2\theta(\lambda) \xrightarrow{\lambda\to\infty} 0 \\ &\text{likelihood that up observed} = E[E_{\rm sup}] = \langle v(\lambda)|E_{\rm sup}v(\lambda)\rangle = |\alpha|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\alpha|^2 \\ &\text{likelihood that dn observed} = E[E_{\rm sdn}] = \langle v(\lambda)|E_{\rm sdn}v(\lambda)\rangle = |\beta|^2\sin^2\theta(\lambda) \xrightarrow{\lambda\to\infty} |\beta|^2. \end{aligned}$$

From (195) and for the state density operator $\rho = |v(\lambda)\rangle\langle v(\lambda)|$ for the pure state $|v(\lambda)\rangle$, the relative state density operators,

$$\rho^{xx} := \frac{E_{xx}\rho E_{xx}}{E[E_{xx}]},$$

are

$$\rho^{\text{no}} = \begin{pmatrix} |\alpha|^2 & 0 & \alpha\overline{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ \overline{\alpha}\beta & 0 & |\beta|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\text{sup}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{\text{sdn}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These are unit trace matrices operating in orthogonal subspaces, that is, with ranges in the null spaces of the other state density operators. $\rho^{xx}\rho^{xx'} = 0$ unless xx'=xx.

References

1. *Handbook of Mathematical Functions*, ed. M. Abramowitz, and I.A. Stegun, National Bureau of Standards, Applied Mathematics Series - 55, 1970.

2. S. Albeverio, H. Gottschalk, and J.-L. Wu, "Convoluted generalized white noise, Schwinger functions and their analytic continuation to Wightman functions," *Rev. Math. Phys.*, Vol. 8, 1996, pg. 763-817.

3. N.I. Akhiezer and, I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, Vol. 1, trans. M. Nestell, New York, NY: Frederick Ungar Publishing Co., 1966.

4. H. Baumgärtel, and M. Wollenberg, "A Class of Nontrivial Weakly Local Massive Wightman Fields with Interpolating Properties," *Commun. Math. Phys.*, Vol. 94, 1984, p. 331.

5. J.S. Bell, "On the Einstein-Podolsky-Rosen Paradox," *Physics*, Vol. 1, 1964, p. 195.

6. G. Birchoff and J. von Neumann, "The Logic of Quantum Mechanics," *The Annals Of Mathematics*, Vol. 37, No. 4, 1936, p. 823.

7. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, New York, NY: McGraw-Hill, 1965.

8. A.S. Blum, "The state is not abolished, it withers away: How quantum field theory became a theory of scattering," *Studies in History and Philosophy of Modern Physics*, Vol. 60, 2017, p. 46.

9. N.N. Bogolubov, A.A. Logunov, and I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, trans. by Stephen Fulling and Ludmilla Popova, Reading, MA: W.A. Benjamin, 1975.

10. H.J. Borchers, "On the structure of the algebra of field operators," *Nuovo Cimento*, Vol. 24, 1962, p. 214.

11. B.S. DeWitt, H. Everett III, N. Graham, J.A. Wheeler in *The Many-worlds Interpretation of Quantum Mechanics*, ed. B.S. DeWitt, N. Graham, Princeton, NJ: Princeton University Press, 1973.

12. J. Dieudonné, *Treatise on Analysis, Volume II*, trans. by I.G. MacDonald, Vol. 10-II in Pure and Applied Mathematics, A Series of Monographs and Textbooks, New York, NY: Academic Press, 1970.

13. P.A.M. Dirac, *The Principles of Quantum Mechanics, Fourth Edition*, Oxford: Clarendon Press, 1958.

14. P. Ehrenfest, "Bemerkung über die angenäherte Gültigkeit der klassichen Machanik innerhalb der Quanatenmechanik," *Z. Physik*, Vol. 45, 1927, p. 455.

15. A. Einstein, "Zur Elektrodynamik bewegter Kórper," *Annalen der Physik* Vol. 322(10), 1905, p. 891.

16. A. Einstein, B. Podolsky and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?," *Physical Review*, Vol. 47(10), 1935, p. 777.

17. Richard Feynman, "Simulating Physics with Computers," *International Journal of Theoretical Physics*, vol. 21, 1982, p. 467.

18. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Volume III*, Reading, MA: Addison-Wesley Publishing Co., 1965.

19. I.M. Gel'fand, and G.E. Shilov, *Generalized Functions, Vol. 1*, trans. E. Salatan, New York, NY: Academic Press, 1964.

20. I.M. Gel'fand, and G.E. Shilov, *Generalized Functions, Vol. 2*, trans. by M.D. Friedman, A. Feinstein, and C.P. Peltzer, New York, NY: Academic Press, 1968.

21. I.M. Gel'fand, and N.Ya. Vilenkin, *Generalized Functions*, Vol. 4, trans. A. Feinstein, New York, NY: Academic Press, 1964.

22. I.M. Gel'fand, M.I. Graev, and N.Ya. Vilenkin, *Generalized Functions*, Vol. 5, trans. E. Saletan, New York, NY: Academic Press, 1966.

23. F. Gross, *Relativistic Quantum Mechanics and Field Theory*, New York, NY: Wiley-Interscience Publication, 1993.

24. B. C. Hall, *Quantum Theory for Mathematicians*, New York: Springer, 2013.

25. H. Georgi, "Unparticle Physics," May 2007, arXiv:hep-ph/0703260.

26. G.C. Hegerfeldt, "Prime Field Decompositions and Infinitely Divisible States on Borchers' Tensor Algebra," *Commun. Math. Phys.*, Vol. 45, 1975, p. 137.

27. R.A. Horn, and C.R. Johnson, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.

28. R.A. Horn, and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge: Cambridge University Press, 1991.

29. J.D. Jackson, *Classical Electrodynamics*, New York, NY: John Wiley and Sons, 1962.

30. A. Jaffe, "Constructive Quantum Field Theory," in *Mathematical Physics 2000*, ed. A. S. Fokas, London: Imperial College Press, 2000.

31. G.E. Johnson, "Consistent Descriptions of Quantum Fields," *Reports on Mathematical Physics*, Vol. 80(2), 2017, p. 193.

32. G.E. Johnson, "Classical approximations of relativistic quantum physics," April 2016, arXiv:quant-ph/1604.07654.

33. G.E. Johnson, "Introduction to quantum field theory exhibiting interaction," Feb. 2015, arXiv:math-ph/1502.07727.

34. G.E. Johnson, "Measurement and self-adjoint operators," May 2014, arXiv:quant-ph/-1405.7224.

35. G.E. Johnson, "Fields and Quantum Mechanics," Dec. 2013, arXiv:math-ph/1312.2608.

36. G.E. Johnson, "Massless Particles in QFT from Algebras without Involution," May 2012, arXiv:math-ph/1205.4323.

37. G.E. Johnson, "Algebras without Involution and Quantum Field Theories," March 2012, arXiv:math-ph/1203.2705.

38. G.E. Johnson, "Are strings the aether of our time?" June 2015, arXiv:quant-ph/1506-.05361v1.

39. G.E. Johnson, "Interacting quantum fields," *Rev. Math. Phys.*, Vol. 11, 1999, p. 881 and errata, *Rev. Math. Phys.*, Vol. 12, 2000, p. 687.

40. R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Volume I: Elementary Theory*, and *Volume II: Advanced Theory*, Graduate Studies in Mathematics, Vol. 15: American Mathematical Society, 1997.

41. G. Lechner, "Deformations of quantum field theories and integrable models," *Commun. Math. Phys.* Vol. 312, 2012, p. 265.

42. A. Messiah, *Quantum Mechanics, Vol. I*, New York, NY: John Wiley and Sons, 1968.

43. T.D. Newton and E.P. Wigner, "Localized States for Elementary Systems," *Rev. Modern Phys.*, Vol. 21, 1949, p. 400.

44. M.F. Pusey, J. Barrett, and T. Rudolph, "On the reality of the quantum state," *Nature Physics*, Vol. 8, 2012, p. 475.

45. H. Reeh and S. Schlieder, "Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern", *Nuovo Cimento*, Vol. 22, 1961, p. 1051.

46. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, New York, NY: Dover Publications, 1990.

47. H. Rinke, "A remark on the asymptotic completeness of local fields," *Commun. Math. Phys.*, Vol. 12, 1969, p. 324.

48. Walter Rudin, *Real & Complex Analysis*, New York, NY: McGraw-Hill Book Co., 1966.

49. E. Schrödinger, "Der stetige Übergang von der Mikro-zur Makromechanik," *Die Naturwissenschaften*, Vol. 14, Issue 28, 1926, p. 664.

50. E. Schrödinger, "Discussion of probability relations between separated systems," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 31 (4), 1935, p. 555.

51. E. Schrödinger, "Die gegenwärtige Situation in der Quantenmechanik," *Die Naturwissenschaften*, Vol. 23, Issue 48, 1935, p. 807.

52. F. Schwabl, *Advanced Quantum Mechanics*, trans. R. Hilton and A. Lahee, Berlin: Springer-Verlag, 1999.

53. I.E. Segal and R.W. Goodman, "Anti-locality of certain Lorentz-invariant operators," *Journal of Mathematics and Mechanics*, Vol. 14, 1965, p. 629.

54. O. Steinmann, "Structure of the Two-Point Function," *J. Math. Phys.*, Vol. 4, 1963, p. 583.

55. A. Douglas Stone, *Einstein and the Quantum*, Princeton, NJ: Princeton University Press, 2013.

56. R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That*, Reading, MA: W.A. Benjamin, 1964.

57. F. Strocchi, "The Physical Principles of Quantum Mechanics. A critical review," Jan. 2012, arXiv:quant-ph/1112.1507.

58. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford: The Oxford University Press, 1948.

59. D. Tong, *String Theory*, *http* : *//www.damtp.cam.ac.uk/user/tong/string/string.pdf*, 2009.

60. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton, NJ: Princeton University Press, 1955.

61. S. Weinberg, *The Quantum Theory of Fields, Volume I, Foundations*, New York, NY: Cambridge University Press, 1995.

62. A.S. Wightman, "Quantum Field Theory in Terms of Vacuum Expectation Values," *Phys. Rev.*, vol. 101, 1956, p. 860.

63. A.S. Wightman and L. Gårding, "Fields as operator-valued distributions in relativistic quantum theory," *Arkiv för Fysik*, Vol. 28, 1965, p. 129.

64. A.S. Wightman,"Hilbert's Sixth Problem: Mathematical Treatment of the Axioms of Physics," *Mathematical Development Arising from Hilbert Problems*, ed. by F. E. Browder, *Symposia in Pure Mathematics 28*, Providence, RI: Amer. Math. Soc., 1976, p. 147.

65. E.P. Wigner, "Remarks on the mind-body question," in I.J. Good, *The Scientist Speculates*, Heinemann, London, 1961.

66. J. Yngvason, "Localization and Entanglement in Relativistic Quantum Physics," Jan. 2014, arXiv:quant-ph/1401.2652.

67. J. Yngvason, "The Role of Type III Factors on Quantum Field Theory," *Reports on Mathematical Physics*, Vol. 55(1), 2005, p. 135.