On bound states in relativistic quantum physics

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Preface: This note demonstrates that bound states of the elementary particles are exhibited by the constructed realizations of relativistic quantum physics based upon VEV.

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1 Introduction

The constructed, fully quantum mechanical realizations of relativistic quantum physics [10, 11, 12, 13] exhibit bound states of the elementary particles. In these realizations, interaction is expressed in the vacuum expectation values (VEV) of fields but not in the Hamiltonians. Appropriate states evolve in correspondence with classical physics descriptions, but canonical quantization's correspondence of densely defined Hermitian operators with classical dynamical variables is only approximated. The constructed Hamiltonians satisfy the requirements of relativity but have only continuous spectra. As a consequence, description of quantized bound states is an evident question for these constructions. This note demonstrates that the constructed realizations of relativistic quantum physics include bound states. There are multiple argument state describing functions with centers-of-momentum described as free particles and with localized descriptions for internals. In a selected reference frame, the internals evolve periodically with time like eigenfunctions of a Hamiltonian. These bound states correspond with classical descriptions of composite particles, similarly to the correspondence of the constructed elementary particles with classically described particles [10].

If a classical correspondence applies, then a bound state consists of two identifiable bodies stably coupled together by an attractive potential. More generally, a quantum mechanical bound state consists of a description for the center-of-momentum as a free particle plus a localized description for the internals. Linear combinations of eigenfunctions from eigenspaces within the continuous spectrum of a Hermitian Hilbert space operator describe bound states. This contrasts with nonrelativistic quantum mechanics that describes bound states as eigenfunctions from the discrete spectra of selected Hamiltonian operators. In both instances, the descriptions of internals have localized support. In the constructions, a description of a bound state follows for every: continuous, bounded, absolutely summable ($\mathcal{L}^1$) function over two momenta; and rest mass $m_b < 2m$. Descriptions of nature are included within the constructions, but are not determined by them.

The constructions [10] describe the evolution of Poincaré invariant likelihoods. These likelihoods include the relative frequency of observing states corresponding to classical descriptions. In the constructions, only states with evident classical correspondences, those with localized and isolated support, are interpreted as corresponding to classical particles. Other considerations, beyond the general physical requirements captured in axioms A.1-7 [10], determine the classical correspondences. Analogously to the many distinct possibilities for bound states within $\mathcal{L}^2$ Hilbert space in nonrelativistic quantum mechanics, the Hilbert spaces $\mathbf{H}_{\mathcal{P}}$ include many descriptions of bound states. While scattering amplitudes are determined by the constructed VEV, significiant freedom remains to specify other classical correspondences. These classical correspondences include the finite period transition amplitudes for particle-like states, and bound states. It is anticipated that this dynamical discriminant corresponds with selection of classically described interactions. In nonrelativistic quantum mechanics, particular bound states follow from a choice of Hamiltonian and other classical correspondences are similarly determined [5, 15]. However, the correspondence of construction and classical Hamiltonian is obscured by the exhibition of multiple, distinct classical correspondences in the differing dynamical regimes of one construction: the effective potential for scattering amplitudes is short range and nuclear-like while the evolution of particle-like states includes long range, gravity-like forces. The development in this note establishes the existence of bound states in RQP, but like other classical correspondences, additional considerations identify the particular descriptions of nature.

2 Preliminaries

One example construction from [10] is applied to the demonstration. This selected construction has:

1. one neutral elementary particle of finite mass m
2. the VEV (5) of a single, scalar quantum field
3. bound states of two elementary particles described by functions with point support in time (1) in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$.

Here, bound states of two elementary particles are described by functions over two spacetime arguments with point support in time. These functions $\tilde{\varphi}_2.\tilde{\psi}_2$ over energy-momenta p_1, p_2 derive from functions $\tilde{f}_2.\tilde{g}_2$ over momenta $\mathbf{p}_1, \mathbf{p}_2$,

$$\begin{aligned}\tilde{\varphi}_2(p_1,p_2) &:= (p_{10}+\omega_1)(p_{20}+\omega_2)\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)\\ \tilde{\psi}_2(p_1,p_2) &:= (p_{10}+\omega_1)(p_{20}+\omega_2)\tilde{g}_2(\mathbf{p}_1,\mathbf{p}_2).\end{aligned} \tag{1}$$

This description is in the Fourier transform domain representation in the two-argument subspace of the constructed Hilbert space $\mathbf{H}_{\mathcal{P}}$ from [10]. $\tilde{\varphi}_2(p_1,p_2) \in \mathbf{H}_{\mathcal{P}}$ if $\tilde{f}_2 \in \mathcal{S}(\mathbb{R}^6)$, a tempered function. The arguments of $\tilde{\varphi}_2(p_1,p_2)$ are wavenumbers $p_\nu \in \mathbb{R}^4$, $\nu = 1,2$. The wavenumber $\omega = \omega(\mathbf{p})$ is proportional to an energy on the mass m shell.

$$\begin{aligned}\omega_j &:= \omega(\mathbf{p}_j)\\ &:= \sqrt{\lambda_c^{-2}+\mathbf{p}_j^2}\end{aligned} \tag{2}$$

with

$$\lambda_c := \frac{\hbar}{mc} \tag{3}$$

the reduced Compton wavelength for the finite mass m in the construction of the single, neutral scalar field $\Phi(x_j)$. Spacetime vectors $x := (x_0, \mathbf{x})$ with $x_0 = ct$ and spatial vectors $\mathbf{x} := x, y, z \in$

$\mathbb{R}^3$ are lengths, energy-momenta are designated $p := (p_0, \mathbf{p})$ with $\mathbf{p} = p_x.p_y, p_z \in \mathbb{R}^3$, momentum vectors $\mathbf{P} = \hbar\mathbf{p}$ and $\hbar$ is Planck's constant h divided by 2π. c is the speed of light. Energies are $E := \hbar c p_0$. Multiple arguments are denoted $(p)_n := p_1, p_2, \dots p_n$. Functions $f_n((x)_n)$ in the basis function spaces $\underline{\mathcal{P}}$ underlying $\mathbf{H}_{\mathcal{P}}$ have Fourier transforms. $\tilde{f}_n((p)_n)$ denotes the Fourier transform of $f_n((x)_n)$ with the four dimensional spacetime Fourier transform adopted here and in [10] the evident multiple argument extension of

$$\tilde{\psi}(p) := \int \frac{dx}{(2\pi)^2}\, e^{-ipx}\psi(x) \tag{4}$$

with the Lorentz invariants $px := p_0 ct - \mathbf{p}\cdot\mathbf{x}$ and spacetime volume element $dx := dx_0 d\mathbf{x}$ with $d\mathbf{x} = dxdydz$. Whether x designates a Cartesian spatial component of $\mathbf{x}$ or a spacetime Lorentz vector is resolved by context. The inverse Fourier transforms of functions over energy-momenta describe the spacetime support of states.

The construction of interest has a single species of mass m neutral scalar elementary particle corresponding with a single scalar field $\Phi(x)$. The cluster expansion of the four-point VEV $\widetilde{\mathcal{W}}_{2,2}((p)_4)$ displays the VEV as the sum of a free field contribution ${}^F\widetilde{\mathcal{W}}_{2,2}((p)_4)$ and a connected contribution ${}^C\widetilde{\mathcal{W}}_{2,2}((p)_4)$. The connected contribution introduces interaction. In this scalar field case of interest, the nonzero four-point VEV are

$$\langle\widetilde{\Phi}(p_2)\widetilde{\Phi}(p_1)\Omega|\widetilde{\Phi}(p_3)\widetilde{\Phi}(p_4)\Omega\rangle = {}^F\widetilde{\mathcal{W}}_{2,2}((p)_4) + {}^C\widetilde{\mathcal{W}}_{2,2}((p)_4) \tag{5}$$

with

$$\begin{aligned}
{}^C\widetilde{\mathcal{W}}_{2,2}((p)_4) &:= c_4\,\delta(p_1{+}p_2{+}p_3{+}p_4)\prod_{j=1}^{4}\delta(p_j^2-\lambda_c^{-2})\\
{}^F\widetilde{\mathcal{W}}_{2,2}((p)_4) &:= (\delta(p_1+p_3)\delta(p_2+p_4)+\delta(p_1+p_4)\delta(p_2+p_3))\prod_{j=3}^{4}\delta(p_j^2-\lambda_c^{-2})
\end{aligned}$$

and

$$\delta(p_j^2-\lambda_c^{-2}) = \frac{\delta(p_{j0}-\omega_j)}{2\omega_j} + \frac{\delta(p_{j0}+\omega_j)}{2\omega_j}.$$

The support of the VEV is limited to mass m shells and the zeros of functions $\tilde{\varphi}_2, \tilde{\psi}_2 \in \mathbf{H}_{\mathcal{P}}$ limit the support of states to positive energy mass shells. For functions (1) from $\mathcal{P}(\mathbb{R}^8)$,

$$\delta(p_1^2-\lambda_c^{-2})\delta(p_2^2-\lambda_c^{-2})\tilde{\varphi}_2(p_1,p_2) = \delta(p_{10}-\omega_1)\delta(p_{20}-\omega_2)\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2) \tag{6}$$

from $(p_{j0}+\omega_j)\,\delta(p_{j0}-\omega_j) = 2\omega_j\delta(p_{j0}-\omega_j)$ and $(p_{j0}+\omega_j)\,\delta(p_{j0}+\omega_j) = 0$. Similarly,

$$\delta(p_1^2-\lambda_c^{-2})\delta(p_2^2-\lambda_c^{-2})\tilde{\varphi}_2^*(p_1,p_2) = \delta(p_{10}+\omega_1)\delta(p_{20}+\omega_2)\overline{\tilde{f}_2}(-\mathbf{p}_2,-\mathbf{p}_1)$$

given the $*$-dual sequence $\underline{\varphi}^* \in \underline{\mathcal{S}}$,

$$\widetilde{\varphi_n^*}((p)_n) := \overline{\tilde{\varphi}_n}(-p_n, \ldots - p_1), \tag{7}$$

in this scalar field case. The $*$-dual includes an argument order reversal, argument reflections and $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. For the single, scalar field construction of interest below, the unitary temporal translation operator is

$$\begin{aligned} U(\lambda)\tilde{\varphi}_2((p)_2) &= e^{-i(p_{0,1}+p_{0,2})\lambda}\tilde{\varphi}_2((p)_2) \\ &= e^{-i(\omega_1+\omega_2)\lambda}\tilde{\varphi}_2((p)_2) \end{aligned} \tag{8}$$

in the Fourier transform domain representation of functions in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$. From (2), ω_j designates $\omega(\mathbf{p}_j)$. $\omega(\mathbf{p}_j)$ is the Hamiltonian for each argument $\mathbf{p}_j$ in this construction with a single mass m elementary particle. From (4), the temporal translation of the description of state, the function, is that a translation of the function by $-\lambda$ corresponds to a translation of the fields by λ.

For the VEV (5), the scalar product of two, two-argument functions is denoted

$$\langle \varphi_2 | \psi_2 \rangle = {}^F\mathcal{W}_{2,2}(\varphi_2^* \, \psi_2) + {}^C\mathcal{W}_{2,2}(\varphi_2^* \, \psi_2) \tag{9}$$

with connected and free field VEV contributions (5). For functions (1), the Fourier transform of generalized functions as Parseval's equality and the dual function (7) provide that

$$\begin{aligned} {}^C\mathcal{W}_{2,2}(\varphi_2^* \, \psi_2) &= c_4 \int d(p)_4 \; \delta(p_1 + p_2 - p_3 - p_4) \prod_{j=1}^{4} \delta(p_{j0} - \omega_j) \; \overline{\tilde{f}_2}(\mathbf{p}_1, \mathbf{p}_2) \; \tilde{g}_2(\mathbf{p}_3, \mathbf{p}_4) \\ {}^F\mathcal{W}_{2,2}(\varphi_2^* \, \psi_2) &= \int d(p)_4 \; (\delta(p_1 - p_3)\delta(p_2 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3)) \\ &\qquad \times \prod_{j=3}^{4} \delta(p_{j0} - \omega_j) \, 4\omega_1\omega_2 \, \overline{\tilde{f}_2}(\mathbf{p}_1, \mathbf{p}_2) \; \tilde{g}_2(\mathbf{p}_3, \mathbf{p}_4). \end{aligned}$$

This evaluation of the scalar product results after substitution of (5), reflection of p_1, p_2, application of the signed symmetry of the VEV, substitutions for the mass shell Dirac delta functions $\delta(p_j^2 - \lambda_c^{-2})$ and factors $(p_{j0} + \omega_j)$ from (6), and with the indicated substitutions from (1). In appendix 4.1, the generalized functions in this scalar product (9) are evaluated and produce the Lebesgue summation,

$${}^C\mathcal{W}_{2,2}(\varphi_2^* \, \psi_2) = c_4 \int d(\mathbf{p})_2 \sin\phi_3' d\theta_3' d\phi_3' \; \frac{\rho_1' \hat{\omega}_3 \hat{\omega}_4}{2\omega_1'} \; \overline{\tilde{f}_2}(\mathbf{p}_1, \mathbf{p}_2) \tilde{g}_2(\hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4) \tag{10}$$

for the connected VEV contribution. The values of $\mathbf{p}_4$ and ρ_3 are determined by energy-momentum conservation for the on-mass shell Lorentz vectors p_j, $j = 1, 2, 3, 4$, and the primed and hat coordinates are defined in appendix 4.1. The Euclidean norms of momentum vectors are designated

$$\rho_j^2 := \mathbf{p}_j^2 := \mathbf{p}_j \cdot \mathbf{p}_j. \tag{11}$$

The $\hat{\omega}_j$ designates $\omega(\hat{\mathbf{p}}_j)$ determined by the constrained momenta $\hat{p}_j$ from (51) in appendix 4.1. $\rho_1' = \|\mathbf{p}_1'\|$ and $\omega_1' = \omega(\mathbf{p}_1')$ with

$$p_1' = \Lambda p_1.$$

Λ is a (p_1+p_2)-dependent Lorentz transformation provided in (46) and (47) of appendix 4.1. The free field contribution to the scalar product also results in a Lebesgue summation

$${}^F\mathcal{W}_{2,2}(\varphi_2^* \psi_2) = \int d(\mathbf{p})_2 \; 4\omega_1\omega_2 \, \overline{\tilde{f}_2}(\mathbf{p}_1, \mathbf{p}_2)(\tilde{g}_2(\mathbf{p}_1, \mathbf{p}_2) + \tilde{g}_2(\mathbf{p}_2, \mathbf{p}_1)). \tag{12}$$

3 Bound states

The description of bound states in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$ develops appropriately supported and temporally evolving functions. The temporal evolution of the center-of-momentum of the bound complex is described as a free particle of mass $m_b < 2m$, and the temporal evolution of the internals is described as if it were an energy eigenfunction with eigenvalue E. The Hamiltonian for evolution of the combined description of the center-of-momentum and internals of the bound state is (8), and this evolution will be assumed to be equivalent to a description of the center-of-momentum as a free particle with a periodic evolution of the internals in a selected reference frame. The description of the internals is required to be spatially localized. A demonstration of this equivalence provides that the support of the description of the center-of-momentum will spread with time consistently with the Heisenberg uncertainty principle. The support of the description of the bound state remains localized over time.

In this section, it is established that bound states exist in the constructed realizations of RQP.

3.1 Relativistic description of bound states

In this study limited to the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$, *bound states* are described by functions $\tilde{\varphi}_2(p_1, p_2)$ of the form (1) with

$$\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2) := \tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2) \, \tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2). \tag{13}$$

The internals of the bound state are described by the function $\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ and $\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)$ independently describes the center-of-momentum of the bound complex. With relativity, the evolution of the internal description of a bound state is coupled to the center-of-momentum description by the Hamiltonian $H = \omega_1 + \omega_2$. This Hamiltonian H does not decompose as a sum of functions over $p_1 + p_2$ and $p_1 - p_2$. In nonrelativistic approximation, this decoupling occurs,

$$\begin{aligned} \omega_1 + \omega_2 &\approx 2\lambda_c^{-1} + \frac{\lambda_c}{2}(\mathbf{p}_1^2 + \mathbf{p}_2^2) \\ &= 2\lambda_c^{-1} + \frac{\lambda_c}{4}(\mathbf{p}_1 + \mathbf{p}_2)^2 + \frac{\lambda_c}{4}(\mathbf{p}_1 - \mathbf{p}_2)^2 \end{aligned}$$

if $\mathbf{p}_1^2$ and $\mathbf{p}_2^2 \ll \lambda_c^{-2}$. In a classical description when relativity is considered, Lorentz-Fitzgerald contraction of body locations varies with the velocity of the center-of-momentum. Thus, $\tilde{f}_2$ includes $\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ rather than a function solely over $\mathbf{p}_1 - \mathbf{p}_2$.

The goal now is to identify functions $\tilde{u}_2$ with localized spatial support. The centers-of-momentum are described by $\tilde{h}_1$. This description for a bound state is expressed using the time translation operator (8) by associating factors with commuting operators and requiring equality of the two state evolution descriptions. A function $\tilde{u}_2$ describes the internals of a bound state if for every multiplier function $\tilde{h}_1$ and all $\lambda \geq 0$,

$$\begin{aligned} U(\lambda)\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\,\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2) &= \left(e^{-i\omega_b(\mathbf{p}_1+\mathbf{p}_2)\,\lambda}\,\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\right)\left(e^{-iE\lambda}\,\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)\right) \\ &= e^{-i(\omega_b(\mathbf{p}_1+\mathbf{p}_2)+E)\lambda}\,\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\,\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2) \end{aligned} \tag{14}$$

with

$$\omega_b(\mathbf{p}) := \sqrt{\lambda_b^{-2} + \mathbf{p}^2} \tag{15}$$

similarly to (2) but with

$$\lambda_b = \frac{\hbar}{m_b c},$$

the reduced Compton wavelength (3) for the bound state mass $m_b < 2m$. The energy E is associated with temporal evolution of the internals in the bound state. The free particle Hamiltonian ω_b describes the evolution of the description of the center-of-momentum, $\tilde{h}_1$. The evolution of an energy eigenfunction, $e^{iE\lambda}$, describes the evolution of $\tilde{u}_2$. These time translation operators mutually commute and commute with functions in the Fourier transform domain. A consequence of cluster decomposition A.6 [10], this description for a bound state applies when the support of $\tilde{f}_2$ in (13) is isolated.

From appendix 4.4, a correspondence with classical particles provides that if the bound state consists of two elementary particles of mass m, then the rest mass of the bound state is $0 \leq m_b < 2m$. The minimal energy that must be added to the bound state to produce two unbound elementary particles is identified as the *binding energy.* This binding energy is the difference of the rest mass energy of two free particles and the bound complex: binding energy $= 2mc^2 - m_b c^2 > 0$. $m_b < 2m$ provides that

$$\lambda_b^{-1} < 2\lambda_c^{-1} \tag{16}$$

from (3) and (15), and then $\lambda_c < 2\lambda_b$. Identifying E as the binding energy sets

$$E = 2\lambda_c^{-1} - \lambda_b^{-1}, \tag{17}$$

the binding energy in wavenumber units.

$$2\lambda_c^{-1} - \lambda_b^{-1} = (2m - m_b)\,\frac{c}{\hbar}.$$

From (14), a locally supported function $\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ describes a bound state if

$$\| |U(\lambda)\tilde{\varphi}_2\rangle - |e^{-i(\omega_b(\mathbf{p}_1+\mathbf{p}_2)+E)\lambda}\,\tilde{\varphi}_2\rangle \| = 0,$$

that due to the unitarity of $U(\lambda)$ is equivalent to

$$\| |U_B\tilde{\varphi}_2\rangle - |\tilde{\varphi}_2\rangle \| = 0. \tag{18}$$

The equivalence (18) is in the Hilbert space $\mathbf{H}_{\mathcal{P}}$ norm with $\tilde{\varphi}_2(p_1, p_2)$ described by (1) and $\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2)$ described by (13). The operator

$$\begin{aligned} U_B &= U_B(\lambda, \mathbf{p}_1, \mathbf{p}_2, E) \\ &:= e^{i(\omega_b(\mathbf{p}_1+\mathbf{p}_2)+E)\,\lambda}\, U(\lambda) \\ &= e^{i(\omega_b(\mathbf{p}_1+\mathbf{p}_2)+E-\omega_1-\omega_2)\,\lambda} \\ &:= e^{-iH_B\lambda} \end{aligned} \tag{19}$$

after substitution of $U(\lambda)$ from (8) and with $\omega_b(\mathbf{p})$ from (15). Noting that U_B is a function of $\mathbf{p}_1+\mathbf{p}_2$ and $\omega_1+\omega_2$, conservation of energy-momentum, $\omega_1+\omega_2 = \omega_3+\omega_4$ and $\mathbf{p}_1+\mathbf{p}_2 = \mathbf{p}_3+\mathbf{p}_4$, provides that $U_B^* = \overline{U_B} = U_B^{-1}$ using the scalar product (9). U_B is unitary. The generator for U_B is defined by (19) as

$$\begin{aligned} H_B &= H_B(\mathbf{p}_1, \mathbf{p}_2, E) \\ &:= \omega_1 + \omega_2 - \omega_b(\mathbf{p}_1 + \mathbf{p}_2) - E \end{aligned} \tag{20}$$

using (2) and (15).

The descriptions (1) of two particle bound states with functions

$$\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2)$$

from (13) satisfy

B.1 the center-of-momentum evolves as a free particle and the internal description remains bound over time: (18) is satisfied for any λ

B.2 the spatial support of the bound state, the support of the Fourier transform of the function $\tilde{u}_2$ in (13), is localized for all λ

B.3 $\tilde{\varphi}_2$ is approximated arbitrarily well by elements within $\mathbf{H}_{\mathcal{P}}$.

A demonstration of B.1-3 establishes that bound states are included in the constructed realizations of relativistic quantum physics. *Localized* includes essentially localized: the descriptions of the constructed states [10, 11] are anti-local functions [19]. B.3 encompases generalized eigenfunctions similarly to eigenfunctions of momentum and location.

Satisfaction of the equality of time evolutions (18) is implied if

$$\langle \varphi_2 | U_B \varphi_2 \rangle = \langle \varphi_2 | \varphi_2 \rangle. \tag{21}$$

Then, a convenient, sufficient condition to satisfy (18) is the eigenvalue problem

$$U_B\, \tilde{\varphi}_2(p_1, p_2) := \tilde{\varphi}_2(p_1, p_2). \tag{22}$$

Satisfaction of (18) derives from (22) and the unitarity of U_B.

$$\langle U_B \varphi_2 | U_B \varphi_2 \rangle = \langle \varphi_2 | \varphi_2 \rangle,$$

and

$$\begin{aligned} 0 &= 2\langle \tilde{\varphi}_2 | \tilde{\varphi}_2 \rangle - 2\langle \tilde{\varphi}_2 | \tilde{\varphi}_2 \rangle) \\ &= \langle U_B \tilde{\varphi}_2 | U_B \tilde{\varphi}_2 \rangle + \langle \tilde{\varphi}_2 | \tilde{\varphi}_2 \rangle - \langle \tilde{\varphi}_2 | U_B \tilde{\varphi}_2 \rangle - \langle U_B \tilde{\varphi}_2 | \tilde{\varphi}_2 \rangle \\ &= \| |U_B \tilde{\varphi}_2\rangle - |\tilde{\varphi}_2\rangle \|^2. \end{aligned}$$

From Stone's theorem [8], the unitarily implemented, one parameter group $U_B(\lambda)$ is generated by a densely defined Hermitian H_B. The eigenvalues of U_B are $\exp(i\mu\lambda)$ with $\mu \in \mathbb{R}$, an eigenvalue of H_B. The bound states $\tilde{\varphi}_2$ are in the null space of H_B.

The eigenfunctions of U_B group into eigenspaces, subspaces generated by eigenfunctions with the same eigenvalue. These eigenspaces include sets of plane waves, generalized functions with point support on pairs of momenta $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^6$, such that

$$H_B(\mathbf{p}_1, \mathbf{p}_2) = \mu.$$

A function

$$\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2) := \delta(H_B(\mathbf{p}_1, \mathbf{p}_2) - \mu)\tilde{\psi}_{u,2}(\mathbf{p}_1, \mathbf{p}_2) \tag{23}$$

is an eigenfunction of U_B from (19) with eigenvalue $\exp(-i\mu\lambda)$. If $\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ is such an eigenfunction, then $\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ is also an eigenfunction with eigenvalue $\exp(-i\mu\lambda)$. The commutation of U_B with functions and (23) provide that $\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ is an eigenfunction: the delta function determines the eigenvalue. Then, $\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)$ is a solution to (22) with minimal volume spatial support within a family of functions $\tilde{f}_2(\mathbf{p}_1,\mathbf{p}_2)$ in (13): $\tilde{u}_2$ is the description of a bound state without a contribution to spreading from a description of the center-of-momentum $\tilde{h}_1 \neq 1$. $\tilde{h}_1(\mathbf{p}) = 1$ exhibits the minimal spatial spread of the support of $\tilde{f}_2$ since the spatial support of f_2 is a convolution of the supports of h_1 and u_2. This description of bound state persists over time λ: the characterization (23) applies for any λ.

The functions (23) with $\mu = 0$ are studied in section 3.2 to select functions of localized spacetime support.

3.2 The spatial support of $\tilde{u}_2$

To satisfy B.2, the inverse Fourier transform of the functions $\tilde{u}_2$ in (23) must be perceived as localized. There are several characterizations of the support of functions in relativistic quantum physics [10, 11, 16]. Here, the inverse Fourier transform u_2 of $\tilde{u}_2$ is used to characterize the spatial support of the state described by φ_2. The support of φ_2 given u_2 is

$$\varphi_2(x_1,x_2) = 2\pi \prod_{j=1}^{2} \left(\delta(x_{j0})\sqrt{-\Delta_j + \lambda_c^{-2}} - i\delta'(x_{j0}) \right) u_2(\mathbf{x}_1,\mathbf{x}_2)$$

from (1) and (4) with Δ_j the Laplacian for $\mathbf{x}_j \in \mathbb{R}^3$. $\sqrt{-\Delta + \lambda_c^{-2}}$ is an anti-local operator [19] and as a consequence, φ_2 is only essentially localized even if the support of u_2 were strictly local [10]. The spatial support of u_2 is characterized by the dominant support over $\mathbf{x}_1, \mathbf{x}_2$ of the inverse Fourier transform of (23).

$$u_2(\mathbf{x}_1,\mathbf{x}_2) := \int d(\mathbf{p})_2 \, \frac{e^{i\mathbf{p}_1\cdot\mathbf{x}_1} e^{i\mathbf{p}_2\cdot\mathbf{x}_2}}{(2\pi)^3} \, \delta(H_B(\mathbf{p}_1,\mathbf{p}_2)) \, \tilde{\psi}_{u,2}(\mathbf{p}_1,\mathbf{p}_2) \tag{24}$$

with a $\tilde{\psi}_{u,2}$ of our choice.

To characterize the support of u_2, the effort now focuses on estimation of the rate of decline of (24) for large $\|\mathbf{x}_j\|$.

The zeros of H_B are required to evaluate (24). From the definition (20) for the generator of U_B and the evaluation (17) of E,

$$\begin{aligned} H_B &= \omega_1 + \omega_2 - \omega_b(\mathbf{p}_1 + \mathbf{p}_2) - E \\ &= \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} - \sqrt{\lambda_b^{-2} + \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos\phi_{12}} + \lambda_b^{-1} - 2\lambda_c^{-1} \end{aligned} \tag{25}$$

with

$$\mathbf{p}_1 \cdot \mathbf{p}_2 := \rho_1\rho_2\cos\phi_{12},$$

the Euclidean norms of the momentum vectors ρ_j from (11), and ϕ_{12} is the angular separation of the momentum vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. H_B is a function over rotational invariants ρ_1, ρ_2 and

$\mathbf{p}_1 \cdot \mathbf{p}_2$. The continuous H_B monotonically decreases with increasing $\cos\phi_{12}$ if $\rho_1\rho_2 \neq 0$. The maximum of H_B occurs at $\mathbf{p}_1 \cdot \mathbf{p}_2 = -\rho_1\rho_2$ and the minimum occurs at $\mathbf{p}_1 \cdot \mathbf{p}_2 = \rho_1\rho_2$. Designate these extrema

$$\begin{aligned} H_{max} &= \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} - \sqrt{\lambda_b^{-2} + (\rho_1 - \rho_2)^2} + \lambda_b^{-1} - 2\lambda_c^{-1} \\ H_{min} &= \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} - \sqrt{\lambda_b^{-2} + (\rho_1 + \rho_2)^2} + \lambda_b^{-1} - 2\lambda_c^{-1}. \end{aligned} \tag{26}$$

If $H_{max} > 0$ and $0 > H_{min}$, the monotonically decreasing, continuous H_B has a single zero for some $\rho_1\rho_2 \cos\phi_{12}$ from the intermediate value theorem. Otherwise, the points $(\mathbf{p})_2 \in \mathbb{R}^6$ that result in both $H_{max}, H_{min} > 0$ or both $H_{max}, H_{min} < 0$ are excluded from the summation (24) characterizing the support of u_2: the delta function is zero at those $(\mathbf{p})_2$.

From (25) and with the designation

$$A := \sqrt{\lambda_c^{-2} + \rho_1^2} + \lambda_b^{-1} - 2\lambda_c^{-1},$$

values of ρ_2 that set $H_B = 0$ are solutions to

$$A + \sqrt{\lambda_c^{-2} + \rho_2^2} = \sqrt{\lambda_b^{-2} + \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos\phi_{12}}.$$

Squaring the equal quantities and reorganization results in

$$2A\sqrt{\lambda_c^{-2} + \rho_2^2} = \lambda_b^{-2} + \rho_1^2 + 2\rho_1\rho_2\cos\phi_{12} - A^2 - \lambda_c^{-2}.$$

Squaring again results in a quadratic polynomial in ρ_2. As a consequence, there are at most two nonnegative real values ρ_2 that set $H_B = 0$. Designate the two solutions to the quadratic polynomial

$$\rho_{o\ell} := \rho_{o\ell}(\rho_1, \phi_{12}) \qquad \ell = 1, 2. \tag{27}$$

Define

$$a_{o\ell}(\rho_1, \phi_{12}) := \begin{cases} 1 \text{ if } \rho_{o\ell} \text{ is real, } \rho_{o\ell} > 0 \text{ and } H_B(\mathbf{p}_1, \mathbf{p}_2) = 0 \text{ at } \rho_1, \rho_{o\ell}, \phi_{12} \\ 0 \text{ otherwise.} \end{cases} \tag{28}$$

This $a_{o\ell}$ includes none, one, or two nonnegative real $\rho_{o\ell}$ for each ρ_1, ϕ_{12} in the summation (24). Only $(\mathbf{p})_2$ that result in real, nonnegative $\rho_{o\ell}$ that set $H_B = 0$ for the given ρ_1, ϕ_{12} are included. The included points are demarcated by Heaviside functions $\theta(H_{max})\theta(-H_{min})$. Given ρ_1, ρ_2, there is exactly one $\cos\phi_{12}$ that sets $H_B = 0$ if $H_{max} > 0$ and $0 > H_{min}$, and no zero otherwise but an $H_{max} > 0$ and $0 > H_{min}$ test is not adequate to eliminate roots introduced by squaring: additional roots are introduced by squaring but these roots apply for distinct ϕ_{12}.

The eigenvalue setting delta function determines whether a $(\mathbf{p})_2$ is included in the summation (24) and determines the up to two ρ_1, ϕ_{12}-specific values for ρ_2 that set $H_B = 0$.

$$\delta(H_B(\mathbf{p}_1, \mathbf{p}_2)) = \sum_{\ell=1}^{2} a_{o\ell}(\rho_1, \phi_{12}) \, \frac{\delta(\rho_2 - \rho_{o\ell})}{\left|\frac{\partial H_B}{\partial \rho_2}\right|}.$$

with

$$\frac{\partial H_B}{\partial \rho_2} = \frac{\rho_2}{\omega_2} - \frac{\rho_2 + \rho_1 \cos\phi_{12}}{\omega_b(\mathbf{p}_1 + \mathbf{p}_2)} \tag{29}$$

from (20). The angular separation ϕ_{12} of momentum vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ has

$$\begin{aligned}
\cos\phi_{12} &= \cos\theta_1 \cos\theta_2 \sin\phi_1 \sin\phi_2 + \sin\theta_1 \sin\theta_2 \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \\
&= \cos(\theta_1 - \theta_2) \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \\
\sin^2\phi_{12} &= \sin^2(\theta_1 - \theta_2) \sin^2\phi_1 \sin^2\phi_2 - 2\cos(\theta_1 - \theta_2) \sin\phi_1 \sin\phi_2 \cos\phi_1 \cos\phi_2 \\
&\quad + \sin^2\phi_1 \cos^2\phi_2 + \cos^2\phi_1 \sin^2\phi_2
\end{aligned} \tag{30}$$

and $\sin\phi_{12} > 0$ in the spherical coordinates

$$\mathbf{p}_j = \begin{pmatrix} \rho_j \cos\theta_j \sin\phi_j \\ \rho_j \sin\theta_j \sin\phi_j \\ \rho_j \cos\phi_j \end{pmatrix}. \tag{31}$$

These coordinates are discussed further below (44) in appendix 4.1. Substitution for $\delta(H_B)$ results in

$$u_2(\mathbf{x}_1, \mathbf{x}_2) = \int d(\mathbf{p})_2 \, \frac{e^{i\mathbf{p}_1 \cdot \mathbf{x}_1} e^{i\mathbf{p}_2 \cdot \mathbf{x}_2}}{(2\pi)^3} \left(\sum_{\ell=1}^{2} a_{o\ell}(\rho_1, \phi_{12}) \, \frac{\delta(\rho_2 - \rho_{o\ell})}{\left|\frac{\partial H_B}{\partial \rho_2}\right|} \right) \tilde{\psi}_{u,2}(\mathbf{p}_1, \mathbf{p}_2). \tag{32}$$

The region included in the summation (32), the points $\mathbf{p}_1, \mathbf{p}_2$ with $H_{max} > 0$ and $0 > H_{min}$, is explored in appendix 4.2 and this region includes points $\mathbf{p}_1, \mathbf{p}_2$:

1. with both $\rho_1, \rho_2 \gg \lambda_c^{-1}$ sufficiently large
2. with $\rho_1 = \rho_2 \neq 0$
3. conditionally, for small $\rho_1, \rho_2 \ll \lambda_b^{-1}$, dependent on the masses m, m_b and the magnitudes of the momenta ρ_1, ρ_2
4. conditionally for $\rho_1 \neq \rho_2$ with the magnitude of one momentum very small with respect to the other.

General properties of the function $\tilde{\psi}_{u,2}(\mathbf{x}_1,\mathbf{x}_2)$ are used to demonstrate that there are bound states within the constructed Hilbert spaces $\mathbf{H}_{\mathcal{P}}$. An absolutely Lebesgue summable function [17] satisfying constraints to eliminate singularities from determination of the eigenspace suffices to imply a bound state. Functions

$$\tilde{\psi}_{u,2}(\mathbf{p}_1,\mathbf{p}_2) = \left(\frac{\partial H_B}{\partial \rho_1}\frac{\partial H_B}{\partial \rho_2}\right)\ \tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2) \tag{33}$$

with $\tilde{\psi}_{o,2} \in \mathcal{L}^1(\mathbb{R}^6)$, pointwise bounded in amplitude and continuous, suffice to characterize bound states. The mollifier functions $\partial H_B/\partial \rho_j$ are summable and bounded by constants. For spherically symmetric bound states, $\tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2)$ depends only on rotational invariants $\|\mathbf{p}_1\|$, $\|\mathbf{p}_2\|$ and $\mathbf{p}_1\cdot\mathbf{p}_2$. Also, anticipate that the description of bound states are transpositionally symmetric in this boson instance,

$$\tilde{\psi}_{u,2}(\mathbf{p}_1,\mathbf{p}_2) = \tilde{\psi}_{u,2}(\mathbf{p}_2,\mathbf{p}_1). \tag{34}$$

The mollifier factors in (33) are bounded in absolute value by a constant and therefore do not affect the absolute summability of $\tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2)$. From (29),

$$\begin{aligned}
\left|\frac{\partial H_B}{\partial \rho_2}\right| &\le \left|\frac{\rho_2}{\omega_2}\right| + \left|\frac{\rho_2+\rho_1\cos\phi_{12}}{\omega_b(\mathbf{p}_1+\mathbf{p}_2)}\right| \\
&\le 1 + \left|\frac{\rho_2+\rho_1\cos\phi_{12}}{\sqrt{\rho_1^2+\rho_2^2+2\rho_1\rho_2\cos\phi_{12}}}\right| \\
&\le 1 + \left|\frac{\rho_2+\rho_1\cos\phi_{12}}{\sqrt{(\rho_2+\rho_1\cos\phi_{12})^2}}\right| \\
&\le 2
\end{aligned} \tag{35}$$

from monotonicity, $\sqrt{a^2+b^2} > \sqrt{c^2+b^2}$ if $a > c$.

Substitution of the mollified function (33) into (32), and a change in summation variables to spherical coordinates (31) results in

$$\begin{aligned}
u_2(\mathbf{x}_1,\mathbf{x}_2) &= \int d(\mathbf{p})_2\, \frac{e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{i\mathbf{p}_2\cdot\mathbf{x}_2}}{(2\pi)^3}\ \left(\sum_{\ell=1}^{2} a_{o\ell}(\rho_1,\phi_{12})\ \delta(\rho_2-\rho_{o\ell})\right) \\
&\qquad \times \tfrac{\partial H_B}{\partial \rho_1}\, \mathrm{sgn}(\tfrac{\partial H_B}{\partial \rho_2})\, \tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2) \\
&= \int_0^\infty \rho_1^2 d\rho_1 \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\phi_1 d\phi_1 \int_0^{2\pi} d\theta_2 \int_0^\pi \sin\phi_2 d\phi_2 \left(\sum_{\ell=1}^{2} a_{o\ell}(\rho_1,\phi_{12})\right. \\
&\qquad \times \left.\rho_{o\ell}^2\, \frac{e^{i\rho_1 r_1\cos\phi_1}e^{i\rho_{o\ell} r_2\cos\phi_2}}{(2\pi)^3}\, \tfrac{\partial H_B}{\partial \rho_1}\, \mathrm{sgn}(\tfrac{\partial H_B}{\partial \rho_2})\, \tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2)\right|_{\rho_2=\rho_{o\ell}}
\end{aligned} \tag{36}$$

for the description of the internals of a bound state. The sign function is $\text{sgn}(x) := x/|x| = \pm 1$. The selection of z-axes aligned with $\mathbf{x}_1$ and $\mathbf{x}_2$ results in

$$\mathbf{p}_j \cdot \mathbf{x}_j = \rho_j r_j \cos\phi_j$$

with $r_j = \|\mathbf{x}_j\|$.

The Riemann-Lebesgue lemma provides that Fourier transforms of Lebesgue summable functions decline to zero for large $\|\mathbf{x}_1\|, \|\mathbf{x}_2\|$. The rate of decay is governed by the differentiablity of the transformed function. The condition (23) that restricts the summation (32) to one eigenspace introduces discontinuities. Localized support is demonstrated by characterizing the rate of decay of (36) with r_j.

Isolating consideration on the ϕ_1 summation in (36), with

$$F_\ell(\rho_1, \theta_1, \theta_2, \phi_2) := \int_0^\pi \sin\phi_1 d\phi_1 \; e^{i\rho_1 r_1 \cos\phi_1} I_\ell(\rho_1, \theta_1, \phi_1, \theta_2, \phi_2) \tag{37}$$

and

$$I_\ell(\rho_1, \theta_1, \phi_1, \theta_2, \phi_2) := e^{i\rho_{o\ell} r_2 \cos\phi_2} \; a_{o\ell}(\rho_1, \phi_{12}) \frac{\rho_{o\ell}^2}{(2\pi)^3} \frac{\partial H_B}{\partial \rho_1} \text{sgn}(\frac{\partial H_B}{\partial \rho_2}) \, \tilde{\psi}_{o,2}(\mathbf{p}_1, \mathbf{p}_2)\bigg|_{\rho_2 = \rho_{o\ell}}$$

then

$$u_2(\mathbf{x}_1, \mathbf{x}_2) = \sum_{\ell=1}^{2} \int_0^\infty \rho_1^2 d\rho_1 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\pi \sin\phi_2 d\phi_2 \; F_\ell(\rho_1, \theta_1, \theta_2, \phi_2).$$

I_ℓ is a piecewise continuous, bounded $\mathcal{L}^1$ function over ϕ_1 given $\rho_1, \theta_1, \theta_2, \phi_2$. The r_2 dependence of I_ℓ, F_ℓ is suppressed in this notation. Substitution of the summation variable $s = \cos\phi_1$, $\sin\phi_1 = \sqrt{1-s^2}$, results in

$$\begin{aligned} r_1 F_\ell(\rho_1, \theta_1, \theta_2, \phi_2) &= r_1 \int_{-1}^{1} ds \; e^{i\rho_1 r_1 s} \; I_\ell \\ &= -\frac{i}{\rho_1} \int_{-1}^{1} ds \; \frac{\partial e^{i\rho_1 r_1 s}}{\partial s} \; I_\ell \end{aligned}$$

The ϕ_j dependence of I_ℓ is $\cos\phi_j$ and $\sin\phi_j$ in ϕ_{12} from (30). The $\mathbf{p}_1 \cdot \mathbf{p}_2$ from H_B in (20) and derivatives (29), and in rotationally invariant $\tilde{\psi}_{o,2}$ from (33) vary with $\cos\phi_{12}$. Labeling the limits of the summation and the discontinuities by a_k results in

$$\begin{aligned} r_1 F_\ell(\rho_1, \theta_1, \theta_2, \phi_2) &= -\frac{i}{\rho_1} \sum_{k=1}^{N} \int_{a_k}^{a_{k+1}} ds \; \frac{\partial e^{i\rho_1 r_1 s}}{\partial s} \; I_\ell \\ &= -\frac{i}{\rho_1} \left(\sum_{k=1}^{N} e^{i\rho_1 r_1 s} I_\ell \bigg|_{a_k}^{a_{k+1}} - \int_{a_k}^{a_{k+1}} ds \; e^{i\rho_1 r_1 s} \; \frac{\partial I_\ell}{\partial s} \right) \end{aligned} \tag{38}$$

with $a_1 = -1$, $a_{N+1} = 1$ and $a_k < a_{k+1}$. The second line results from integration by parts. Upper bounding the amplitude of the sum by the sum of amplitudes results in

$$r_1|F_\ell(\rho_1,\theta_1,\theta_2,\phi_2)| \leq \frac{1}{\rho_1}\left(\sum_{k=1}^{N}|I_\ell|\Big|_{a_k}^{a_{k+1}} + \int_{a_k}^{a_{k+1}} ds\, \left|\frac{\partial I_\ell}{\partial s}\right|\right).$$

This upper bound is independent of both r_1 and r_2. The ρ_1^{-1} times the ρ_1^2 from the Jacobian to spherical coordinates leaves the summable ρ_1.

Concerns are whether I_ℓ has summable divergences that become unsummable with differentiation. This is not the case. The function I_ℓ is bounded, continuous and piecewise differentiable. Discontinuities occur in $a_{o\ell}(\rho_1,\phi_{12})$ from (28) and in $\mathrm{sgn}(\partial H_B/\partial\rho_2)$ from (29). $\rho_{o\ell}$ from (27), $\partial H_B/\partial\rho_1$ from (29) and our choice of $\tilde{\psi}_{o,2}(\mathbf{p}_1,\mathbf{p}_2)$ are once differentiable as summable functions over s from the chain rule.

$$\frac{d\sqrt{1-s^2}}{ds} = \frac{s}{\sqrt{1-s^2}} = \frac{s}{\sqrt{(1-s)(1+s)}}.$$

The roots $\rho_{o\ell}$ of the quadratic polynomial vary continuously with the coefficients [9] and the coefficients are differentiable. Diverging roots[a] $\rho_{o\ell}$ are controlled by the rapid decline of $\tilde{\psi}_{o\ell}$. Exclusion of complex roots and roots introduced by squaring is implemented by $a_{o\ell}(\rho_1,\phi_{12})$. The function $\partial I_\ell/\partial s$ is an absolutely summable function of ϕ_1 and its summation produces a $\mathcal{L}^1$ function over $\rho_1,\theta_1,\theta_2,\phi_2$

Second or higher derivatives of $e^{i\rho_1 r_1 s}$ with the discontinuous I_ℓ result in derivatives of delta functions [6, 14] that restore factors of r_1 and consequently do not result in demonstrations of a faster rate of decline with r_1.

Substitution of (38) into (37), upper bounding $r_1 u_2(\mathbf{x}_1,\mathbf{x}_2)$ with absolute values and absolute summability provides that

$$r_1|u_2(\mathbf{x}_1,\mathbf{x}_2)| \leq A.$$

Transpositional symmetry provides a similar decline with ρ_2.

$$|u_2(\mathbf{x}_1,\mathbf{x}_2)| \leq \min_{r_1,r_2}\left(\frac{A}{r_1},\frac{A}{r_2}\right). \tag{39}$$

The bound (39) suffices to demonstrate that the dominant support of $u_2(\mathbf{x}_1,\mathbf{x}_2)$ from (24) lies within finite spheres centered on the bound state. For many functions $\tilde{\psi}_{u,2}(\mathbf{p}_1,\mathbf{p}_2)$, the likelihood per unit volume for an observation of location decreases with expanding distance r_1, r_2 from the center of support.

[a] With the quadratic polynomial designated $a\rho_2^2 + b\rho_2 + c = 0$, a divergent root occurs for $a \to 0$.

The spatial support of u_2 characterizes the likelihoods of location measurements. From Born's rule, likelihoods follow from the scalar product (9),

$$\begin{aligned}\text{Likelihood} &= \frac{\langle\varphi_2|P_{\mathbf{x}_1,\mathbf{x}_2}\varphi_2\rangle|^2}{\langle\varphi_2|\varphi_2\rangle} \\ &= \frac{|\langle\psi_2|\varphi_2\rangle|^2}{\langle\varphi_2|\varphi_2\rangle\langle\psi_2|\psi_2\rangle}\end{aligned}$$

with $P_{\mathbf{x}_1,\mathbf{x}_2}$ the projection operator onto states with support limited to $\mathbf{x}_1,\mathbf{x}_2$,

$$\tilde{g}_2(\mathbf{p_1},\mathbf{p}_2) := \frac{e^{-i\mathbf{p}_1\cdot\mathbf{x}_1}e^{-i\mathbf{p}_2\cdot\mathbf{x}_2}}{(2\pi)^3}$$

and $\tilde{f}_2 := \tilde{u}_2$. In (1), $\tilde{f}_2, \tilde{g}_2$ define $\tilde{\varphi}_2, \tilde{\psi}_2$. This likelihood is a limit from a sequence of elements in $\mathbf{H}_{\mathcal{P}}$ approaching the eigenfunction of location with eigenvalues $\mathbf{x}_1, \mathbf{x}_2$. Eigenstates of location are idealized and not elements of the $\mathcal{L}^2$, Fock, nor $\mathbf{H}_{\mathcal{P}}$ Hilbert spaces used in quantum mechanics.

The free field VEV contribution to this likelihood is evaluated using (12),

$${}^F\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2) = \int \frac{d(\mathbf{p})_2}{(2\pi)^3}\, 8\omega_1\omega_2\, e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{i\mathbf{p}_2\cdot\mathbf{x}_2}\tilde{u}_2(\mathbf{p}_1,\mathbf{p}_2)$$

for argument transposition symmetric functions, and the connected VEV contribution is from (10),

$$\begin{aligned}{}^C\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2) &= c_4\int \frac{d(\mathbf{p})_2}{(2\pi)^3}\sin\phi_3' d\theta_3' d\phi_3'\; \frac{\rho_1'\hat{\omega}_3\hat{\omega}_4}{2\omega_1'}\; e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{i\mathbf{p}_2\cdot\mathbf{x}_2}\tilde{u}_2(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4) \\ &= c_4\int \sin\phi_3' d\theta_3' d\phi_3' \left(\int \frac{d(\mathbf{p})_2}{(2\pi)^3}\; \frac{\rho_1'\hat{\omega}_3\hat{\omega}_4}{2\omega_1'}\; e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{i\mathbf{p}_2\cdot\mathbf{x}_2}\tilde{u}_2(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4)\right).\end{aligned}$$

Then, Laplacians Δ_j operating on u_2 provide the free field VEV contribution to the likelihood of location. From appendix 4.1, the constrained momenta $\hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4$ are the spatial components of rotations, boosts and energy-momentum rescales followed by the inverse boosts and rotations.

$$\hat{p}_j = \mathcal{R}^{-1}\mathcal{B}^{-1}C(\mathcal{B}\mathcal{R}p_j)$$

for rotations $\mathcal{R}$, pure boosts $\mathcal{B}$ and the energy-momentum rescaling $C(p)$ determined by $\mathbf{p}_1, \mathbf{p}_2$, appendix 4.1. Up to a rotation, the constrained momenta are determined by the boost and rescale,

$$\mathcal{R}\hat{p}_j = \mathcal{B}^{-1}C(\mathcal{B}\mathcal{R}p_j),$$

and then if $\tilde{u}_2$ is a function solely of the rotational invariants $\|\mathbf{p}_1\|$, $\|\mathbf{p}_2\|$ and $\mathbf{p}_1 \cdot \mathbf{p}_2$,

$$\tilde{u}_2(\hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4) \approx \tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2),$$

in nonrelativistic instances, those with $\mathcal{B} \approx 1$. Except for relativistic distortion, finite summations operated upon by Laplacians of an approximation with rotationally invariant weightings ρ_1'/ω_1' within the summations appear in the connected VEV contribution to the likelihood of location. And, although the individual rotational invariants $\|\mathbf{p}_1\|$, $\|\mathbf{p}_2\|$ and $\mathbf{p}_1 \cdot \mathbf{p}_2$ distort in relativistic instances, conservation of energy-momentum, $p_1 + p_2 = \hat{p}_3 + \hat{p}_4$, provides that their sum is relativistically invariant,

$$\|\hat{\mathbf{p}}_3 + \hat{\mathbf{p}}_4\|^2 = \|\mathbf{p}_1 + \mathbf{p}_2\|^2 = \|\mathbf{p}_1\|^2 + \|\mathbf{p}_2\|^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2.$$

Then, the support of u_2 characterizes the likelihood of location.

3.3 Approximation of $\tilde{u}_2$ within $\mathbf{H}_{\mathcal{P}}$

In this section it is demonstrated that the description of a bound state from section 3 is not an element of the Hilbert space $\mathbf{H}_{\mathcal{P}}$, but is arbitrarily well approximated by elements. This satisfies B.3. The functions $\tilde{\varphi}_2(p_1, p_2)$ from (1), (13) and (23) have infinite norm, but like momentum and location eigenfunctions, are arbitrarily well approximated by elements of $\mathbf{H}_{\mathcal{P}}$.

The free field contribution to the scalar product is the Lebesgue summation (12),

$$\begin{aligned} {}^F\mathcal{W}_{2,2}(\varphi_2^* \varphi_2) &= \int d(\mathbf{p})_2\, 4\omega_1\omega_2\, \overline{\tilde{u}_2}(\mathbf{p}_1, \mathbf{p}_2)(\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2) + \tilde{u}_2(\mathbf{p}_2, \mathbf{p}_1)) \\ &= \int d(\mathbf{p})_2\, 8\omega_1\omega_2\, |\tilde{\psi}_{u,2}(\mathbf{p}_1, \mathbf{p}_2)|^2\, \delta^2(H_B(\mathbf{p}_1, \mathbf{p}_2)). \end{aligned} \tag{40}$$

from the transpositional symmetry of H_B in (20) and $\tilde{\psi}_{u,2}$ in (34). Then, if the four-point VEV includes this free field contribution, the norm $\|\tilde{\varphi}_2\|$ diverges: $\delta^2(x)$ is divergent. In this evaluation $\tilde{h}_1 = 1$, but introduction of another description $\tilde{h}_1$ for the center-of-momentum that is a multiplier of test functions does not affect this argument. The functions $\tilde{\varphi}_2(\mathbf{p}_1, \mathbf{p}_2)$ that describe bound states are not elements of $\mathbf{H}_{\mathcal{P}}$. This result is similar to that the eigenfunctions of momentum and location are not elements of the Hilbert spaces for the relativistic quantum physics constructions [10], for relativistic free fields [3], nor for the $\mathcal{L}^2$ Hilbert spaces of nonrelativistic quantum mechanics. And, like the location and momentum eigenfunctions, the functions $\tilde{\varphi}_2(\mathbf{p}_1, \mathbf{p}_2)$ are arbitrarily well approximated by Hilbert space elements. For illustration, functions (23) with a delta sequence of Gaussian functions substituted for $\delta(H_B(\mathbf{p}_1, \mathbf{p}_2)$ are elements of $\mathbf{H}_{\mathcal{P}}$.

The connected VEV contribution to the scalar product is the Lebesgue summation (10),

$$\begin{aligned}{}^{C}\mathcal{W}_{2,2}(\varphi_2^*\varphi_2) &= c_4\int d(\mathbf{p})_2 \sin\phi_3' d\theta_3' d\phi_3' \,\frac{\rho_1'\hat{\omega}_3\hat{\omega}_4}{2\omega_1'}\,\overline{\tilde{u}_2}(\mathbf{p}_1,\mathbf{p}_2)\tilde{u}_2(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4)\\ &= c_4\int d(\mathbf{p})_2 \sin\phi_3' d\theta_3' d\phi_3' \,\frac{\rho_1'\hat{\omega}_3\hat{\omega}_4}{2\omega_1'}\,\overline{\tilde{\psi}_{u,2}}(\mathbf{p}_1,\mathbf{p}_2)\tilde{\psi}_{u,2}(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4)\\ &\qquad\times\delta(H_B(\mathbf{p}_1,\mathbf{p}_2)\,\delta(H_B(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4)\end{aligned} \tag{41}$$

with $\tilde{h}_1 = 1$, and the summations over $\mathbf{p}_4$ and ρ_3 determined by energy-momentum conservation for the four on-mass shell Lorentz vectors p_j, $j = 1, 2, 3, 4$. The energy-momentum conservation-constrained momentum components $\hat{p}_j$ from (51) in appendix 4.1 are functions over $\mathbf{p}_1, \mathbf{p}_2$ and θ_3', ϕ_3'. $\rho_1' = \|\mathbf{p}_1'\|$ and $\omega_1' = \omega(\mathbf{p}_1')$ with

$$p_1' = \Lambda p_1.$$

Λ is a (p_1+p_2)-dependent Lorentz transformation provided in (46) and (47) of appendix 4.1. From momentum conservation,

$$(\hat{\mathbf{p}}_3 + \hat{\mathbf{p}}_4)^2 = (\mathbf{p}_1 + \mathbf{p}_2)^2$$

and the conservation of energy,

$$\omega_1 + \omega_2 = \hat{\omega}_3 + \hat{\omega}_4,$$

The evaluation (20) of H_B provides

$$\begin{aligned} H_B(\mathbf{p}_1,\mathbf{p}_2) &= \omega_1 + \omega_2 - \sqrt{\lambda_b^{-2} + (\mathbf{p}_1+\mathbf{p}_2)^2} + \lambda_b^{-1} - 2\lambda_c^{-1}\\ &= \hat{\omega}_3 + \hat{\omega}_4 - \sqrt{\lambda_b^{-2} + (\hat{\mathbf{p}}_3+\hat{\mathbf{p}}_4)^2} + \lambda_b^{-1} - 2\lambda_c^{-1}\\ &= H_B(\hat{\mathbf{p}}_3,\hat{\mathbf{p}}_4).\end{aligned}$$

As a consequence, there is a factor of $\delta^2(H_B(\mathbf{p}_1, \mathbf{p}_2)$ in the evaluation of the connected VEV contribution to the scalar product and this contribution to the norm diverges similarly to the free field VEV contribution.

3.4 Properties of relativistic bound states

In this section it is demonstrated that bound states are orthogonal to all plane wave states except those with zero momenta. As a consequence, the likelihood of producing a bound state in a scattering event vanishes unless the VEV include connected functions of the same order as the VEV.

Two classically described particles cannot scatter into a bound state without additional products to carry away excess energy. This result applies to the relativistic quantum description of bound states. Below, the demonstration includes that $\langle \mathbf{q}_1, \mathbf{q}_2 | \tilde{\varphi}_2 \rangle = 0$: the two-argument bound states are orthogonal to plane wave states of the elementary particles unless $\mathbf{q}_1 = \mathbf{q}_2 = 0$.

The existence of bound states did not require any special properties of the VEV, but the demonstration below illustrates that bound states do not result from collisions of the free elementary particles unless the constructed VEV include connected contributions.

From (9) and for a state description $\tilde{\psi}_2$ that is an eigenfunction of two momenta, the scattering amplitude for two incoming (near) plane wave states described by $\mathbf{q}_1, \mathbf{q}_2$ transitioning to an outgoing state that consists solely of a bound state described by $\tilde{u}_2$ is

$$\text{Scattering amplitude} = \langle \psi_2 | \varphi_2 \rangle$$

with

$$\tilde{g}_2(\mathbf{p_1}, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{q}_1)\delta(\mathbf{p}_2 - \mathbf{q}_2),$$

$\tilde{f}_2 := \tilde{h}_1 \tilde{u}_2$ from (13) and $\tilde{u}_2$ is from (23). In (1), $\tilde{f}_2, \tilde{g}_2$ define $\tilde{\varphi}_2, \tilde{\psi}_2$. This amplitude is a limit of a sequence of elements in $\mathbf{H}_{\mathcal{P}}$ approaching the eigenfunction of momenta with eigenvalues $\mathbf{q}_1, \mathbf{q}_2$. Momentum eigenfunctions are idealizations and not elements of the Hilbert spaces $\mathcal{L}^2$, Fock space, nor $\mathbf{H}_{\mathcal{P}}$ used in quantum mechanics. The free field contribution to this scalar product is (12),

$$\begin{aligned} {}^F\mathcal{W}_{2,2}(\psi_2^* \varphi_2) &= \int d(\mathbf{p})_2 \, 8\omega_1\omega_2 \, \delta(\mathbf{p}_1 - \mathbf{q}_1)\delta(\mathbf{p}_2 - \mathbf{q}_2)\tilde{f}_2(\mathbf{p}_1, \mathbf{p}_2) \\ &= 8\omega(\mathbf{q}_1)\omega(\mathbf{q}_2)\, \tilde{f}_2(\mathbf{q}_1, \mathbf{q}_2) \end{aligned}$$

for argument transposition symmetric functions, and the connected VEV contribution to the scalar product is (10),

$$\begin{aligned} {}^C\mathcal{W}_{2,2}(\psi_2^* \varphi_2) &= c_4 \int d(\mathbf{p})_2 \sin\phi_3' d\theta_3' d\phi_3' \, \frac{\rho_1' \hat{\omega}_3 \hat{\omega}_4}{2\omega_1'} \, \delta(\mathbf{p}_1 - \mathbf{q}_1)\delta(\mathbf{p}_2 - \mathbf{q}_2)\tilde{f}_2(\hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4) \\ &= c_4 \int \sin\phi_3' d\theta_3' d\phi_3' \, \frac{\|\mathbf{q}_1'\| \hat{\omega}_3 \hat{\omega}_4}{2\omega(\mathbf{q}_1')} \, \tilde{f}_2(\hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4). \end{aligned}$$

The primed and hatted momentum variables $\mathbf{p}_j'$ and $\hat{\mathbf{p}}_j$ are described in appendix 4.1 as linear combinations

$$p' = \Lambda p$$

and momentum magnitude adjusted vectors with Λ defined from $\mathbf{q}_1 + \mathbf{q}_2$.

In the single, scalar field instance, an automorphism of $\mathcal{P}$ implements Poincaré transformations.

$$(a, \Lambda)\underline{\varphi} := (\varphi_o, \ldots \varphi_n(\Lambda^{-1}(x_1 - a), \ldots \Lambda^{-1}(x_n - a)), \ldots)$$

with Λ a proper orthochronous Lorentz transformation and a a Lorentz vector translation. The automorphism is

$$(a,\Lambda)\underline{\tilde{\varphi}} := (\varphi_o, \dots \exp(i(p_1+p_2\dots+p_n)a)\tilde{\varphi}_n(\Lambda^{-1}p_1,\dots\Lambda^{-1}p_n),\dots)$$

in the Fourier transform domain. The descriptions of states are Poincaré covariant

$$\varphi_2 \mapsto (a,\Lambda)\varphi_2$$

and the scalar product (9) is Poincaré invariant.

$$\langle\psi_2|\varphi_2\rangle = \langle(a,\Lambda)\psi_2|(a,\Lambda)\varphi_2\rangle.$$

The Poincaré invariance of amplitudes provides that the amplitude can be evaluated in the center-of-momentum frame for $\mathbf{q}_1, \mathbf{q}_2$ without loss of generality. In this frame, the rest frame of $q_1 + q_2$,

$$\mathbf{q}_1 + \mathbf{q}_2 = 0$$

and from the description of bound states (23), the functions $\tilde{f}_2$ include a factor

$$\delta(H_B(\mathbf{p}_1, \mathbf{p}_2))$$

that is zero for both ${}^F\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$ and ${}^C\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$ unless $\mathbf{q}_1 = \mathbf{q}_2 = 0$. This result follows from inspection of H_B in the center-of-momentum frame. From conservation of energy-momentum, $p_1 + p_2 = \hat{p}_3 + \hat{p}_4$ and then

$$\mathbf{q}_1 + \mathbf{q}_2 = \hat{\mathbf{p}}_3 + \hat{\mathbf{p}}_4 = 0.$$

In the evaluation of ${}^F\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$ or ${}^C\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$, the definition (20) of H_B and the evaluation (17) of E provides that if $\mathbf{q}_1 + \mathbf{q}_2 = 0$ then

$$\begin{aligned} H_B &= \omega_1 + \omega_2 - \omega_b(\mathbf{q}_1 + \mathbf{q}_2) - E \\ &= \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} - 2\lambda_c^{-1} \\ &\geq 0 \end{aligned}$$

in the center-of-momentum frame. H_B is strictly greater than zero unless $\rho_1 = \rho_2 = 0$. ρ_1, ρ_2 designates $\|\mathbf{q}_1\|, \|\mathbf{q}_2\|$ in the instance of ${}^F\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$ and $\|\hat{\rho}_3\|, \|\hat{\rho}_4\|$ in the instance of ${}^C\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2)$. As a consequence, the bound states are orthogonal to the plane wave descriptions of free elementary particles,

$$\langle\psi_2|\varphi_2\rangle := \langle\mathbf{q}_1, \mathbf{q}_2|\varphi_2\rangle = 0$$

for all nonzero $\mathbf{q}_1, \mathbf{q}_2$. The orthogonality applies for any time since the time translates of the momentum eigenfunctions are equal up to a phase.

$$U(\lambda)|\mathbf{q}_1, \mathbf{q}_2\rangle = e^{-i(\omega_1+\omega_2)\lambda}|\mathbf{q}_1, \mathbf{q}_2\rangle$$

from (8).

The discussion of bound states within the two-argument subspace did not assume any properties of the VEV. However, with $\langle \mathbf{q}_1, \mathbf{q}_2|\varphi_2\rangle = 0$, if the free field VEV were all that appeared in the development, then no collision could ever produce a bound state. For example, if a ${}^C\mathcal{W}_{3,3}$ were excluded, then the cluster decomposition of the VEV displays the scalar product composed of only lower order VEV. These all vanish for incoming states described as plane waves with outgoing states consisting of plane waves and bound states. With three incoming elementary particles transitioning to a bound state plus one free elementary particle, the description of a two-argument bound state equates

$$\begin{aligned} U(\lambda)\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\,\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)\tilde{v}_1(\mathbf{p}_3) & \\ = \left(e^{-i\omega_b(\mathbf{p}_1+\mathbf{p}_2)\,\lambda}\,\tilde{h}_1(\mathbf{p}_1 + \mathbf{p}_2)\right) & \left(e^{-iE\lambda}\,\tilde{u}_2(\mathbf{p}_1, \mathbf{p}_2)\right)\left(e^{-i\omega_3\lambda}\tilde{v}_1(\mathbf{p}_3)\right) \end{aligned}$$

with $\tilde{v}_1$ a description of an asymptotically free elementary particle. In these higher order VEV instances, the development of $\tilde{u}_2$ is the same as in section 3.1. However, with the additional arguments, conservation of energy-momentum no longer implies the equality of subsets of momenta,

$$0 = \mathbf{q}_1 + \mathbf{q}_2 \neq \hat{\mathbf{p}}_3 + \hat{\mathbf{p}}_4.$$

As a consequence, amplitudes such as

$$\langle \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3|\varphi_2, \mathbf{q}_4\rangle$$

do not generally vanish if a connected function ${}^C\mathcal{W}_{3,3}$ contributes. If the VEV are free field VEV, then momenta are equal in pairs and there are no transitions from plane wave states to bound states in scattering amplitudes.[b]

Bound states are created from collisions of free elementary particles if higher order connected functions contribute. Existence of bound state did not require knowledge of the VEV, but the likelihood of creating a bound state is zero without connected contributions to VEV.

[b] In the general construction of relativistic quantum mechanics, the elementary particles are produced or annihilated in pairs: only even order VEV appear [10]. However, with the scalar representation of the Poincaré group, odd order VEV may appear [11].

4 Appendices

4.1 Energy conservation

The connected VEV (5) include both energy-momentum conservation and on-mass shell Dirac delta functions. The summations in the scalar product (9) are constrained to manifolds within $(p)_4 \in \mathbb{R}^{16}$ within the support of

$$\delta(p_1 + p_2 - p_3 - p_4) \prod_{j=1}^{4} \delta(p_j^2 - \lambda_c^{-2}).$$

Within this manifold, the energy and momenta of four on-mass shell m Lorentz vectors p_j add to zero. The joint support of the VEV (5) and functions (1) from $\mathcal{P}$ provides that each energy of a state describing functions is on a positive energy mass m shell, $p_{j0} = \omega_j$ from (2).

$$\mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3$$

satisfies the momentum conservation constraint. Satisfaction of the remaining energy support condition is considered a constraint on $\mathbf{p}_3$. This constraint is evaluated in this appendix. Evaluation of the generalized function that implements energy conservation leaves a Lebesgue summation for the scalar product (9) in the two-argument subspace of $\mathbf{H}_{\mathcal{P}}$.

The scalar product within the two-argument subspace (9) derives from the VEV (5) for functions $\psi_2, \varphi_2 \in \mathcal{P}(\mathbb{R}^8)$ from (1). The connected component of the scalar product $\langle \psi_2 | \varphi_2 \rangle$ from (9) exhibits interaction and equals

$${}^{C}\mathcal{W}_{2,2}(\psi_2^* \, \varphi_2) = c_4 \int d(p)_4 \; \delta(p_1{+}p_2{-}p_3{-}p_4) \prod_{j=1}^{4} \delta(p_{j0} - \omega_j) \; \overline{\tilde{g}_2}(p_2, p_1) \; \tilde{f}_2(p_3, p_4)$$

with the Lorentz invariants $\delta(p)$ and

$$\delta(p_j^2 - \lambda_c^{-2}) \, \theta(p_{j0}) = \frac{\delta(p_{j0} - \omega_j)}{2\omega_j}.$$

After evaluation of the mass shell and momentum conservation delta functions, the argument of the energy conservation delta function is

$$\omega_1{+}\omega_2{-}\omega_3{-}\omega_4 = \omega(\mathbf{p}_1){+}\omega(\mathbf{p}_2){-}\omega(\mathbf{p}_3){-}\omega(\mathbf{p}_1{+}\mathbf{p}_2{-}\mathbf{p}_3) \tag{42}$$

using (2). (42) is zero if $\mathbf{p}_3 = \mathbf{p}_1$ or $\mathbf{p}_2$, the forward scatter cases. More generally, (42) is zero if $\mathbf{p}_3$, is constrained within a $\mathbf{p}_1, \mathbf{p}_2$-dependent manifold.

A change of summation variables,

$$p'_j := \Lambda p_j \tag{43}$$

for $j = 1, 2, 3, 4$, is selected to transform to the center-of-momentum frame for each p_1, p_2. The transformation is a proper (det(Λ)=1), orthochronous ($\Lambda_{00} > 0$) Lorentz transformation

$$\Lambda := \mathcal{BR}$$

consisting of a rotation $\mathcal{R}$ and a boost $\mathcal{B}$. To evaluate Λ, designate the center-of-momentum for each p_1, p_2

$$q := p_1 + p_2 = \begin{pmatrix} \omega_1 + \omega_2 \\ \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix} \tag{44}$$

in polar coordinates with the momentum $\mathbf{q} := (q_x, q_y, q_z)$,

$$\rho := \|\mathbf{q}\| = \|\mathbf{p}_1 + \mathbf{p}_2\|,$$

and

$$\begin{aligned} \cos\phi &= \frac{q_z}{\rho}, & \sin\phi &= \frac{\sqrt{q_x^2 + q_y^2}}{\rho} \\ \cos\theta &= \frac{q_x}{\sqrt{q_x^2 + q_y^2}}, & \sin\theta &= \frac{q_y}{\sqrt{q_x^2 + q_y^2}} \\ \cos\theta\sin\phi &= \frac{q_x}{\rho}, & \sin\theta\sin\phi &= \frac{q_y}{\rho} \end{aligned} \tag{45}$$

with quadrants selected for θ, ϕ to correspond with the signs of q_x, q_y, q_z. $\theta \in \{0, 2\pi\}$ is the anticlockwise angle of $\mathbf{q}$ from the x-axis in the x-y plane, and $\phi \in \{0, \pi\}$ is the angle of $\mathbf{q}$ from the z-axis in the plane containing $\mathbf{q}$ and the z-axis. Then, the rotation $\mathcal{R}$ aligns the momentum $\mathbf{q}$ with the primed z-axis,

$$\begin{pmatrix} \omega_1 + \omega_2 \\ 0 \\ 0 \\ \rho \end{pmatrix} = \mathcal{R}q := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ 0 & \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \omega_1 + \omega_2 \\ \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix} \tag{46}$$

and the boost $\mathcal{B}$ is along the rotated z-axis and zeros the momentum.

$$\begin{pmatrix} 2\omega_1' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathcal{BR}q := \begin{pmatrix} \lambda_c\omega(\beta) & 0 & 0 & -\lambda_c\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -0 \\ -\lambda_c\beta & 0 & 0 & \lambda_c\omega(\beta) \end{pmatrix} \begin{pmatrix} \omega_1 + \omega_2 \\ 0 \\ 0 \\ \rho \end{pmatrix} \tag{47}$$

with $\omega(\beta)$ determined similarly to (2),

$$\omega(\beta) = \sqrt{\lambda_c^{-2} + \beta^2} = \lambda_c^{-1}\sqrt{1 + \left(\frac{\hbar\beta}{mc}\right)^2},$$

and

$$-\lambda_c\beta\,(\omega_1 + \omega_2) + \lambda_c\omega(\beta)\rho = 0.$$

Then,

$$\lambda_c\beta = \frac{\rho}{\sqrt{(\omega_1 + \omega_2)^2 - \rho^2}}. \tag{48}$$

β is nonnegative and real since $\omega_1 + \omega_2 > \rho$ for finite m with $\rho = \|\mathbf{p}_1 + \mathbf{p}_2\|$. λ_c is the reduced Compton's wavelength (3). In this primed, center-of-momentum frame, $\omega'_1 = \omega'_2$. Both the rotation $\mathcal{R}$ and boost $\mathcal{B}$ are determined by $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2$ and (2). This transformation can alternatively be parameterized

$$\omega(\beta) = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)^{-1},$$

relating velocity v to the momentum β. Rotations are orthogonal transformations and the inverse of the rotation is $\mathcal{R}^{-1} = \mathcal{R}^T$. The inverse of the boost is $\mathcal{B}$ evaluated with the negative momentum, $\beta \mapsto -\beta$.

Satisfaction of momentum conservation is evident in the primed, center-of-momentum frame of reference. In this frame and for each particular $\mathbf{u}$, $\mathbf{p}'_2 = -\mathbf{p}'_1$ and then momentum conservation provides that $\mathbf{p}'_4 = -\mathbf{p}'_3$. With this change to summation variables (43), the connected contribution to the scalar product (9) becomes

$$\begin{aligned}
{}^{C}\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2) &= c_4 \int d(p')_4\;\delta(p'_1+p'_2-p'_3-p'_4)\prod_{j=1}^{4}\delta(p_{j0}-\omega_j)\\
&\qquad\times\overline{\tilde{g}_2}(\Lambda^{-1}p'_2,\Lambda^{-1}p'_1)\tilde{f}_2(\Lambda^{-1}p'_3,\Lambda^{-1}p'_4)\\
&= c_4 \int d(p')_4\;\delta(2\omega'_1-2\omega'_3)\;\delta(\mathbf{p}'_3+\mathbf{p}'_4)\prod_{j=1}^{4}\delta(p_{j0}-\omega_j)\\
&\qquad\times\overline{\tilde{g}_2}(\Lambda^{-1}p'_2,\Lambda^{-1}p'_1)\;\tilde{f}_2(\Lambda^{-1}p'_3,\Lambda^{-1}p'_4)
\end{aligned}$$

noting the Lorentz invariance of $\delta(p) = \delta(\Lambda p)$ from $|\det(\Lambda)| = 1$. The energy conservation delta function is

$$\begin{aligned}
\delta(2\omega'_3 - 2\omega'_1) &= \frac{\delta(\rho'_3 - \rho'_1)}{2\left|\dfrac{d\omega(\mathbf{p}'_3)}{d\rho'_3}\right|}\\
&= \frac{\omega'_1}{2\rho'_1}\,\delta(\rho'_3 - \rho'_1)
\end{aligned} \tag{49}$$

[6] from $\rho'_3 = \rho'_1$ with $\rho'_j := \|\mathbf{p}'_j\|$ and $\omega'_j := \omega(\mathbf{p}'_j)$.

If the primed, unconstrained

$$p'_3 := \begin{pmatrix} \omega'_3 \\ \rho'_3 \cos\theta'_3 \sin\phi'_3 \\ \rho'_3 \sin\theta'_3 \sin\phi'_3 \\ \rho'_3 \cos\phi'_3 \end{pmatrix} \qquad \text{and} \qquad p'_4 = \vartheta p'_3$$

then the primed, constrained

$$\hat{p}'_3 := \begin{pmatrix} \omega'_1 \\ \rho'_1 \cos\theta'_3 \sin\phi'_3 \\ \rho'_1 \sin\theta'_3 \sin\phi'_3 \\ \rho'_1 \cos\phi'_3 \end{pmatrix} \qquad \text{and} \qquad \hat{p}'_4 = \vartheta \hat{p}'_3 \tag{50}$$

with the spatial reflection ϑ defined

$$\vartheta\hat{p}'_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \hat{p}'_3 = \begin{pmatrix} \omega'_1 \\ -\rho'_1 \cos\theta'_3 \sin\phi'_3 \\ -\rho'_1 \sin\theta'_3 \sin\phi'_3 \\ -\rho'_1 \cos\phi'_3 \vartheta \end{pmatrix}.$$

The polar angles θ'_3, ϕ'_3 are in the primed coordinate frame.

Now, evaluation of the momentum conservation delta function, and substitution of the simplified energy conservation delta function and the constrained $\hat{p}'_3$ provides that

$$\begin{aligned} {}^{C}\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2) &= c_4 \int d(p')_2 dp'_{30} dp'_{40}\, d\Omega'_3\, {\rho'_1}^2\, \frac{\omega'_1}{2\rho'_1} \prod_{j=1}^{4} \delta(p_{j0}-\omega_j) \\ &\qquad \times \overline{\tilde{g}_2}(\Lambda^{-1}p'_2, \Lambda^{-1}p'_1) \tilde{f}_2(\Lambda^{-1}\hat{p}'_3, \Lambda^{-1}\hat{p}'_4). \end{aligned}$$

Ω'_3 are the polar angle coordinates of $\mathbf{p}'_3$, $d\mathbf{p}'_3 = {\rho'_3}^2 d\rho'_3 d\Omega_3$ with $d\Omega'_3 := \sin\phi'_3 d\theta'_3 d\phi'_3$. Note that this summation would diverge in fewer than $2+1$ dimensions for finite m [10, 13].

From the Lorentz invariance of $\delta(p^2 - m^2)\theta(p_0)$,

$$\frac{\delta(p_{j0}-\omega_j)}{2\omega_j} = \frac{\delta(p'_{j0}-\omega'_j)}{2\omega'_j}$$

with Lorentz transformation. In the center-of-momentum frame, $\hat{\omega}'_3 = \hat{\omega}'_4 = \omega'_1$, evaluation of the p_{30}, p_{40} delta functions result in

$$\begin{aligned} {}^{C}\mathcal{W}_{2,2}(\psi_2^*\,\varphi_2) &= c_4 \int d(p')_2\, d\Omega'_3\, {\rho'_1}^2\, \frac{\omega'_1}{2\rho'_1} \prod_{j=1}^{2} \delta(p_{j0}-\omega_j)\, \frac{\hat{\omega}_3\hat{\omega}_4}{\hat{\omega}'_3\hat{\omega}'_4}\, \overline{\tilde{g}_2}\, \tilde{f}_2 \\ &= c_4 \int d(p')_2\, d\Omega'_3\, \frac{\rho'_1\hat{\omega}_3\hat{\omega}_4}{2\omega'_1} \prod_{j=1}^{2} \delta(p_{j0}-\omega_j)\, \overline{\tilde{g}_2}(\Lambda^{-1}p'_2, \Lambda^{-1}p'_1) \tilde{f}_2(\Lambda^{-1}\hat{p}'_3, \Lambda^{-1}\vartheta\hat{p}'_3). \end{aligned}$$

Note that $\hat{\omega}_j$ for $j = 3, 4$ designates $\omega(\hat{\mathbf{p}}_j)$ using (2) determined from the momentum components of the unprimed, constrained values of the momentum components of

$$\hat{p}_3 := \Lambda^{-1}\hat{p}'_3 \qquad \text{and} \qquad \hat{p}_4 := \Lambda^{-1}\vartheta\hat{p}'_3 \tag{51}$$

and (50). $\hat{\omega}'_j$ designates $\omega(\hat{\mathbf{p}}'_j)$. From (43), $\mathbf{p}'_1$ are the spatial components of Λp_1 and $\rho'_1 = \|\mathbf{p}'_1\|$.

Finally, return to the original summation variables p_1, p_2 produces

$$\langle\psi_2|\varphi_2\rangle = c_4 \int d(\mathbf{p})_2 \; d\Omega'_3 \; \frac{\rho'_1 \omega_3 \omega_4}{2\omega'_1} \; \overline{\tilde{g}_2}(p_2, p_1) \tilde{f}_2(\hat{p}_3, \hat{p}_4),$$

an explicit Lebesgue summation for the scalar product (9). Note that each energy-momentum is on a mass m shell, $p_{j0} = \omega_j$ from (2), and (51) for $j = 3, 4$.

Both $\mathcal{B}$ and $\mathcal{R}$ are determined by u from (44), $\mathbf{u} = \mathbf{p}_1 + \mathbf{p}_2$, and from ω_1 and ω_2. ρ'_1 and ω'_1 are determined from $\mathbf{p}'_1$ using (43) and p_1.

$$p'_1 = \Lambda p_1 = \Lambda \begin{pmatrix} \omega_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}.$$

$\hat{p}'_3$ and $\hat{p}'_4 = \vartheta\hat{p}'_3$ are determined from ρ'_1, θ'_3 and ϕ'_3 in (50), and $\hat{p}_3, \hat{p}_4$ are determined in (51).

The Lorentz vector

$$\hat{p}'_3 + \hat{p}'_4 = (2\omega'_1, 0, 0, 0)^T$$

is time-like, and

$$\hat{p}'_3 - \hat{p}'_4 = (0, 2\hat{\mathbf{p}}'_3)$$

is space-like. Proper orthochronous transformation Λ^{-1} of time-like vectors remain time-like, and the transformation of space-like vectors remain space-like. These vectors are also orthogonal in the Minkowski dot product, evident above and since both $\hat{p}_3$ and $\hat{p}_4$ are rest mass m Lorentz energy-momentum vectors,

$$(\hat{p}_3 + \hat{p}_4)(\hat{p}_3 - \hat{p}_4) = \hat{p}_3^2 - \hat{p}_4^2 = m^2 - m^2 = 0.$$

The Lorentz vector

$$\begin{aligned} \hat{p}_3 + \hat{p}_4 &= \Lambda^{-1}(\hat{p}'_3 + \hat{p}'_4) \\ &= \mathcal{R}^{-1}\mathcal{B}^{-1}(2\omega'_1, 0, 0, 0)^T \\ &= 2\lambda_c\omega'_1\mathcal{R}^{-1}(\omega(\beta), 0, 0, \beta)^T \\ &= 2\lambda_c\omega'_1 \begin{pmatrix} \omega(\beta) \\ \beta\cos\theta\sin\phi \\ \beta\sin\theta\sin\phi \\ \beta\cos\phi \end{pmatrix} \end{aligned}$$

independently of Ω_3, in the same direction as $\mathbf{u}$, with magnitude adjusted to conserve energy with on-mass m shell energy-momenta, and with momentum magnitude β. The polar angles θ, ϕ are determined by $\mathbf{p}_1+\mathbf{p}_2$ in (44). From conservation of energy-momentum, $p_1+p_2 = \hat{p}_3+\hat{p}_4$ and then

$$\|\hat{\mathbf{p}}_3 + \hat{\mathbf{p}}_4\| = \|\mathbf{p}_1 + \mathbf{p}_2\| = \rho = 2\lambda_c\beta\omega_1',$$

a succinct relationship of ω_1', ρ and β. Then (48) provides relationships of the prime and unprimed frame quantities

$$\begin{aligned} 2\omega_1' &= \sqrt{(\omega_1+\omega_2)^2 - \rho^2} \\ 2\rho_1' &= \sqrt{(\omega_1+\omega_2)^2 - \rho^2 - 4\lambda_c^{-2}}. \end{aligned}$$

Also,

$$\begin{aligned} \hat{p}_3 - \hat{p}_4 &= \Lambda^{-1}(\hat{p}_3' - \hat{p}_4') \\ &= \mathcal{R}^{-1}\mathcal{B}^{-1}(0, 2\hat{\mathbf{p}}_3')^T \\ &= 2\rho_1'\mathcal{R}^{-1}(\lambda_c\beta\cos\phi_3', \cos\theta_3'\sin\phi_3', \sin\theta_3'\sin\phi_3', \lambda_c\omega(\beta)\cos\phi_3')^T \\ &= 2\rho_1' \begin{pmatrix} \lambda_c\beta\cos\phi_3' \\ \sin\theta\cos\theta_3'\sin\phi_3' + \cos\theta\cos\phi\sin\theta_3'\sin\phi_3' + \lambda_c\omega(\beta)\cos\theta\sin\phi\cos\phi_3' \\ \cos\theta\cos\theta_3'\sin\phi_3' + \sin\theta\cos\phi\sin\theta_3'\sin\phi_3' + \lambda_c\omega(\beta)\sin\theta\sin\phi\cos\phi_3' \\ -\sin\phi\sin\theta_3'\sin\phi_3' + \lambda_c\omega(\beta)\cos\phi\cos\phi_3' \end{pmatrix}. \end{aligned}$$

The primed and original variables are linearly related by (43), $p_j' = \Lambda p_j$.

4.2 The supported region

The support included in the Fourier transform (24) includes the $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^6$ with $H_B(\mathbf{p}_1, \mathbf{p}_2) = 0$. H_B is from (20). This region include momenta $\mathbf{p}_1, \mathbf{p}_2$ with magnitudes ρ_1, ρ_2 (11) such that $H_{max} > 0$ and $0 > H_{min}$. H_{max} and H_{min} are evaluated in (26). The continuous, monotonic $H_{max} > H_B(\mathbf{p}_1, \mathbf{p}_2) > H_{min}$ and then there is a $\mathbf{p}_1 \cdot \mathbf{p}_2$ with $H_B = 0$. In this appendix, this region within $\mathbb{R}^6$ is characterized using estimates for the functions ω_j from (2) and $\omega_b(\mathbf{p})$ from (15).

Insight into the region included in the summation (24) follows from small and large momentum Taylor series approximations to the energies from (2) and (15). For $\mathbf{p}^2 \ll \lambda_c^{-2}$,

$$\sqrt{\lambda_c^{-2} + \mathbf{p}^2} \approx \lambda_c^{-1} + \frac{\lambda_c}{2}\mathbf{p}^2$$

and for $\mathbf{p}^2 \gg \lambda_c^{-2}$,

$$\sqrt{\lambda_c^{-2} + \mathbf{p}^2} \approx \|\mathbf{p}\| + \frac{\lambda_c^{-2}}{2\|\mathbf{p}\|}.$$

Three cases are examined: both momenta large with respect to λ_c^{-1} (relativistic); both momenta small with respect to λ_c^{-1} (nonrelativistic); and one momenta very small with respect to the second momenta that is large with respect λ_c^{-1}.

If the magnitude squared of both momenta are relativistic, magnitudes squared are large with respect to λ_c^{-2}, and if $(\mathbf{p}_1 - \mathbf{p}_2)^2$ is large with respect to λ_b^{-2}, then (26) and the Taylor series approximation provide that

$$\begin{aligned}
H_{max} &\approx \rho_1 + \rho_2 + \frac{\lambda_c^{-2}}{2\rho_1} + \frac{\lambda_c^{-2}}{2\rho_2} - |\rho_1 - \rho_2| - \frac{\lambda_b^{-2}}{2|\rho_1 - \rho_2|} + \lambda_b^{-1} - 2\lambda_c^{-1} \\
&= 2\min(\rho_1, \rho_2) + \left(1 - \frac{\lambda_b^{-1}}{2|\rho_1 - \rho_2|}\right)\lambda_b^{-1} - \left(2 - \frac{\lambda_c^{-1}}{2\rho_1} - \frac{\lambda_c^{-1}}{2\rho_2}\right)\lambda_c^{-1} \\
H_{min} &\approx \rho_1 + \rho_2 + \frac{\lambda_c^{-2}}{2\rho_1} + \frac{\lambda_c^{-2}}{2\rho_2} - \rho_1 - \rho_2 - \frac{\lambda_b^{-2}}{2(\rho_1 + \rho_2)} + \lambda_b^{-1} - 2\lambda_c^{-1} \\
&= \left(1 - \frac{\lambda_b^{-1}}{2(\rho_1 + \rho_2)}\right)\lambda_b^{-1} - \left(2 - \frac{\lambda_c^{-1}}{2\rho_1} - \frac{\lambda_c^{-1}}{2\rho_2}\right)\lambda_c^{-1}.
\end{aligned}$$

$H_{max} > 0$ for sufficiently large ρ_1, ρ_2. Similarly, $H_{min} < 0$ unless

$$0 < 2\lambda_c^{-1} - \lambda_b^{-1} < \frac{\lambda_c^{-2}}{2\rho_1} + \frac{\lambda_c^{-2}}{2\rho_2} - \frac{\lambda_b^{-2}}{2(\rho_1 + \rho_2)}.$$

This never applies for sufficiently large ρ_1, ρ_2 and then $H_{min} < 0$.

The instances with $(\rho_1 - \rho_2)^2 \ll \lambda_b^{-2}$ and relativistic $\rho_1, \rho_2 \gg \lambda_c^{-1}$ result in

$$H_{max} \approx \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} - 2\lambda_c^{-1} - \frac{\lambda_b}{2}(\rho_1 - \rho_2)^2 > 0$$

always satisfied for sufficiently large ρ_1, ρ_2. The bounds on H_{min} do not require $(\rho_1 - \rho_2)^2 \ll \lambda_b^{-2}$ and the previous estimate for relativistic ρ_1, ρ_2 applies.

Then, for both ρ_1, ρ_2 sufficiently large, $H_{max} > 0$ and $0 > H_{min}$.

If the magnitude squared of both momenta are sufficiently nonrelativistic, the magnitudes squared are small with respect to both λ_c^{-2} and λ_b^{-2}, then

$$\begin{aligned}
H_{max} &\approx 2\lambda_c^{-1} + \tfrac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) - \lambda_b^{-1} - \tfrac{\lambda_b}{2}(\rho_1 - \rho_2)^2 + \lambda_b^{-1} - 2\lambda_c^{-1} \\
&= \tfrac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) - \tfrac{\lambda_b}{2}(\rho_1 - \rho_2)^2 \\
H_{min} &\approx 2\lambda_c^{-1} + \tfrac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) - \lambda_b^{-1} - \tfrac{\lambda_b}{2}(\rho_1 + \rho_2)^2 + \lambda_b^{-1} - 2\lambda_c^{-1} \\
&= \tfrac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) - \tfrac{\lambda_b}{2}(\rho_1 + \rho_2)^2.
\end{aligned}$$

H_{min} is less than zero if

$$\frac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) < \frac{\lambda_b}{2}(\rho_1 + \rho_2)^2$$

or, reorganizing,

$$\frac{\lambda_c - \lambda_b}{2}(\rho_1^2 + \rho_2^2) < \lambda_b \rho_1 \rho_2.$$

H_{max} is positive if

$$\frac{\lambda_c}{2}(\rho_1^2 + \rho_2^2) > \frac{\lambda_b}{2}(\rho_1 - \rho_2)^2$$

or, reorganizing,

$$\frac{\lambda_c - \lambda_b}{2}(\rho_1^2 + \rho_2^2) > -\lambda_b \rho_1 \rho_2.$$

Then, the points $\mathbf{p}_1, \mathbf{p}_2$ are included in the summation (24) if

$$-\frac{\rho_1 \rho_2}{\rho_1^2 + \rho_2^2} < \frac{\lambda_c - \lambda_b}{2\lambda_b} < \frac{\rho_1 \rho_2}{\rho_1^2 + \rho_2^2}.$$

If this condition is violated, then $H_{max} \leq 0$ or $H_{min} \geq 0$.

In summary, for sufficiently small ρ_1, ρ_2, $H_{max} > 0$ and $0 > H_{min}$ are conditional, dependent on the mass m_b of the bound state relative to the elementary mass m and the magnitudes of the momenta ρ_1, ρ_2. For sufficiently small ρ_1, ρ_2, there are points ρ_1, ρ_2 excluded from the summation (24). For sufficiently small ρ_1, ρ_2 and $\rho_1 = \rho_2$, the condition becomes

$$0 < \lambda_c < 2\lambda_b$$

always satisfied due to the requirement for a bound state that $\lambda_b^{-1} < 2\lambda_c^{-1}$, appendix 4.4. But, if $\rho_2 = 0$, the condition becomes

$$\lambda_c = \lambda_b.$$

Instances with $\rho_1 = 0$ or $\rho_2 = 0$ are singular cases excluded by the lack of support of the summation (36) on zero momenta. But, by continuity, these simple instances provide insight. For $\rho_1 = 0$ or $\rho_2 = 0$,

$$H_{max} = H_{min} = H_B$$

and there is no zero of H_B unless every point with $\rho_1 = 0$ or $\rho_2 = 0$ is a zero. If $\rho_1 = 0$, then

$$H_{max} = H_{min} = \sqrt{\lambda_c^{-2} + \rho_2^2} - \lambda_c^{-1} - \sqrt{\lambda_b^{-2} + \rho_2^2} + \lambda_b^{-1}$$

from (2), (15) and (26). Then, the monotonic increase of $\omega_2 - \lambda_c^{-1}$ and $\omega_b(\mathbf{p}_2) - \lambda_b^{-1}$ with increasing reciprocal wavelengths provides that both H_{max} and H_{min} are negative for $m_b < m$. Both H_{max} and H_{min} are positive for $m_b > m$. If $m_b = m$, both H_{max} and H_{min} are zero for either $\rho_1 = 0$ or $\rho_2 = 0$.

For the magnitude squared of one momenta very small with respect to the other, taking $\rho_1 \gg \rho_2$, Taylor series expansion results in

$$\begin{aligned} H_{max} &= \sqrt{\lambda_c^{-2}+\rho_1^2}+\sqrt{\lambda_c^{-2}+\rho_2^2}-\sqrt{\lambda_b^{-2}+\rho_1^2}+\frac{\rho_1\rho_2}{\sqrt{\lambda_b^{-2}+\rho_1^2}}+\lambda_b^{-1}-2\lambda_c^{-1} \\ H_{min} &= \sqrt{\lambda_c^{-2}+\rho_1^2}+\sqrt{\lambda_c^{-2}+\rho_2^2}-\sqrt{\lambda_b^{-2}+\rho_1^2}-\frac{\rho_1\rho_2}{\sqrt{\lambda_b^{-2}+\rho_1^2}}+\lambda_b^{-1}-2\lambda_c^{-1}. \end{aligned}$$

and then $H_{max} > 0$ for a sufficienly large product $\rho_1\rho_2$. $0 > H_{min}$ is also conditional and requires that

$$\sqrt{\lambda_c^{-2}+\rho_1^2}+\sqrt{\lambda_c^{-2}+\rho_2^2}-\sqrt{\lambda_b^{-2}+\rho_1^2}+\lambda_b^{-1}-2\lambda_c^{-1} < \frac{\rho_1\rho_2}{\sqrt{\lambda_b^{-2}+\rho_1^2}}.$$

But again, for sufficiently large ρ_1, ρ_2, the leading order in the Taylor series expansions is

$$\lambda_b^{-1} < 2\lambda_c^{-1}$$

that is necessarily satisfied and the next terms in the Taylor series are negligible if ρ_1, ρ_2 are sufficiently large ρ_1, ρ_2.

Another insight into the support of the summation in (32) is for equal magnitude momenta, $\rho_1 = \rho_2$. These cases include $\mathbf{p}_2 = -\mathbf{p}_1$ of interest for the classical correspondence in the center of momentum frame. In these instances,

$$\begin{aligned} H_{max} &= 2\sqrt{\lambda_c^{-2}+\rho_1^2}-2\lambda_c^{-1} \\ H_{min} &= 2\sqrt{\lambda_c^{-2}+\rho_1^2}-2\lambda_c^{-1}-2\sqrt{\tfrac{1}{4}\lambda_b^{-2}+\rho_1^2}+\lambda_b^{-1}. \end{aligned}$$

and these points are always included in the summation in (32). $H_{max} > 0$ follows from $\sqrt{a^2+b^2} > a$ for non-zero, positive real numbers, and $0 > H_{min}$ follows from $H_{min} = 0$ at $\rho_1 = 0$ and the derivative of the continuous H_{min} with respect to ρ_1,

$$\frac{dH_{min}}{d\rho_1} = \frac{2\rho_1}{\sqrt{\lambda_c^{-2}+\rho_1^2}} - \frac{2\rho_1}{\sqrt{\frac{1}{4}\lambda_b^{-2}+\rho_1^2}},$$

is strictly negative for $\rho_1 > 0$. The negativity of the derivative follows from the monotonicity of $\sqrt{a^2+\rho^2}$ in both a and ρ, and that $\lambda_b^{-1} < 2\lambda_c^{-1}$.

4.3 Zeros of H_{min} and H_{max}

It is established in appendix 4.2 that there are values ρ_1, ρ_2 that result in $H_{min} = 0$ or $H_{max} = 0$. At these ρ_1, ρ_2, the continuous functions H_{min} or H_{max} transition from negative to positive values. The values of ρ_1, ρ_2 at the zeros of H_{min} or H_{max} form a curve that demarcates the support of the summation (32). In this appendix, these zeros are evaluated.

From (26), $H_{min} = 0$ is equivalent to

$$\begin{aligned} \sqrt{\lambda_c^{-2} + \rho_1^2} + \sqrt{\lambda_c^{-2} + \rho_2^2} &= \sqrt{\lambda_b^{-2} + (\rho_1 + \rho_2)^2} - \lambda_b^{-1} + 2\lambda_c^{-1} \\ \sqrt{\lambda_c^{-2} + \tfrac{(u+v)^2}{4}} + \sqrt{\lambda_c^{-2} + \tfrac{(u-v)^2}{4}} &= \sqrt{\lambda_b^{-2} + u^2} - \lambda_b^{-1} + 2\lambda_c^{-1} \end{aligned}$$

with the substitutions

$$\begin{aligned} u &= \rho_1 + \rho_2 \\ v &= \rho_1 - \rho_2. \end{aligned}$$

Squaring,

$$\begin{aligned} 2\lambda_c^{-2} + \frac{(u+v)^2}{4} + \frac{(u-v)^2}{4} + 2\sqrt{\lambda_c^{-4} + \lambda_c^{-2}\left(\tfrac{(u+v)^2}{4} + \tfrac{(u-v)^2}{4}\right) + \tfrac{(u+v)^2(u-v)^2}{16}} &= \left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 \\ 2\lambda_c^{-2} + \frac{v^2}{2} + \frac{u^2}{2} + 2\sqrt{\lambda_c^{-4} + \tfrac{\lambda_c^{-2}}{2}(u^2 + v^2) + \tfrac{(u^2-v^2)^2}{16}} &= \left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 \end{aligned}$$

with

$$0 < A := 2\lambda_c^{-1} - \lambda_b^{-1} < 2\lambda_c^{-1}.$$

Reorganizing and squaring again results in a linear equation for v^2.

$$\begin{aligned} 2\sqrt{\lambda_c^{-4} + \tfrac{\lambda_c^{-2}}{2}(u^2 + v^2) + \tfrac{(u^2-v^2)^2}{16}} &= \left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 - 2\lambda_c^{-2} - \frac{u^2}{2} - \frac{v^2}{2} \\ 4\lambda_c^{-4} + 2\lambda_c^{-2}(u^2 + v^2) + \frac{(u^2 - v^2)^2}{4} &= \left(\left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 - 2\lambda_c^{-2} - \frac{u^2}{2} - \frac{v^2}{2}\right)^2 \\ 4\lambda_c^{-4} + 2\lambda_c^{-2}(u^2 + v^2) + \frac{u^4 - 2u^2v^2}{4} &= \left(\left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 - 2\lambda_c^{-2} - \frac{u^2}{2}\right)^2 \\ &\quad - \left(\left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 - 2\lambda_c^{-2} - \frac{u^2}{2}\right) v^2. \end{aligned}$$

The linear equation for v^2 that sets $H_{min} = 0$ results from collecting like powers of v.

$$\begin{aligned} &\left\{\left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 + u^2\right\} v^2 \\ &\quad = \left(\left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2 - 2\lambda_c^{-2} - \frac{u^2}{2}\right)^2 - 4\lambda_c^{-4} - 2\lambda_c^{-2}u^2 - \frac{u^4}{4}. \end{aligned} \tag{52}$$

Then, there are two solutions, designated $v = \pm v_o(u)$, to $H_{min} = 0$. The roots of interest have $\rho_1 > 0$ and $\rho_2 > 0$. If $u > v_o(u)$,

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}(u \pm v_o(u)) \\ \rho_2 &= \tfrac{1}{2}(u \mp v_o(u)) \end{aligned}$$

are the zeros of H_{min}.

(52) results in values for v^2 that are less than, equal, or exceed u^2 as λ_b is varied with respect to λ_c. With

$$B := \left(\sqrt{\lambda_b^{-2} + u^2} + A\right)^2,$$

the value of λ_b that results in $v^2 = u^2$ is verified from (52).

$$u^2 = v^2 = \frac{B^2 - (4\lambda_c^{-2} + u^2)B}{B + u^2}$$

or

$$0 = \frac{B^2 - (4\lambda_c^{-2} + u^2)B - u^2(B + u^2)}{B + u^2}.$$

From $B + u^2 > 0$, $u^2 = v^2$ if

$$0 = B^2 - (4\lambda_c^{-2} + 2u^2)B - u^4.$$

Substitution for B and A results in

$$B = \left(\sqrt{\lambda_b^{-2} + u^2} + 2\lambda_c^{-1} - \lambda_b^{-1}\right)^2 = 2\lambda_c^{-2} + u^2 \pm \sqrt{(2\lambda_c^{-2} + u^2)^2 + u^4}.$$

Only the plus sign provides $B > 0$, and $\lambda_c^{-1} = \lambda_b^{-1}$ results in the identity

$$\left(\sqrt{\lambda_c^{-2} + u^2} + \lambda_c^{-1}\right)^2 = 2\lambda_c^{-2} + u^2 + 2\lambda_c^{-1}\sqrt{\lambda_c^{-2} + u^2}. \tag{53}$$

Then (53) and inspection of (52) result in

$$\begin{aligned} &u^2 > v^2 &&\text{if} &&\lambda_c^{-2} < \lambda_b^{-2} < 2\lambda_c^{-2} \\ &u^2 = v^2 &&\text{if} &&\lambda_b^{-2} = \lambda_c^{-2} \\ &u^2 < v^2 &&\text{if} &&0 < \lambda_b^{-2} < \lambda_c^{-2}. \end{aligned}$$

The zeros of H_{min} follow if $u^2 > v^2$. $u^2 = v^2$ is a singular point with either $\rho_1 = 0$ or $\rho_2 = 0$. In this instance, $m_b = m$ and for this single, scalar field case of present interest, the bound state is indiscernible from an elementary particle. This point is excluded by a lack of support of the summation (36) on zero momenta. If $u^2 < v^2$, then either $\rho_1 < 0$ or $\rho_2 < 0$ and this is equivalent to an exchange of H_{min} with H_{max}. That is, the zero corresponds to changing the sign of the dot product $\mathbf{p}_1 \cdot \mathbf{p}_2$, or taking $-\cos\phi_{12} = \cos(\pi - \phi_{12})$ with $\rho_1, \rho_2 > 0$. These values of ρ_1, ρ_2 are zeros of H_{max} with $H_{min} < 0$. Indeed, from (26) and with substitution of u, v for ρ_1, ρ_2, $H_{max} = 0$ is equivalent to

$$\sqrt{\lambda_c^{-2} + \frac{(u+v)^2}{4}} + \sqrt{\lambda_c^{-2} + \frac{(u-v)^2}{4}} = \sqrt{\lambda_b^{-2} + v^2} - \lambda_b^{-1} + 2\lambda_c^{-1}$$

that exchanges u and v in the development of the zeros of H_{min} above. Finally, the zeros evaluated in (52) are zeros of

$$\begin{aligned} H_{min} \quad &\text{if} \quad \lambda_c^{-2} < \lambda_b^{-2} < 2\lambda_c^{-2} \\ H_{max} \quad &\text{if} \quad 0 < \lambda_b^{-2} < \lambda_c^{-2}. \end{aligned}$$

4.4 Relativistic bound-state kinematics

In the single, finite mass elementary particle instance of present interest, a bound state composed of N elementary particles is described by a center-of-momentum that behaves like a free particle with a rest mass less than Nm, with a localized description for the internal degrees of freedom. From the cluster decomposition property of VEV [10], the bound states and elementary particles have correspondences with classical bodies when the support of the descriptions of the bound states or elementary particles are distantly space-like separated from the support of other bodies. Without describing likelihoods nor the dynamics of state descriptions, the rest masses of bound states are described by the classical correspondences from the results of collisions of elementary particles with the bound states.

If the initial state is described by one freely propagating bound state described by its center-of-momentum and one elementary particle, and the final state is described by some greater number of elementary particles, then collision instances can be selected to evaluate the rest mass of the bound state. With the bound state described by a rest mass m_b, in the center-of-momentum reference frame for the collision, the initial energy-momenta are

$$(\omega_1, \mathbf{p}_1) \qquad \text{and} \qquad (\sqrt{\lambda_b^{-2} + \mathbf{p}_1^2}, -\mathbf{p}_1)$$

with $\lambda_b = \hbar/(m_b c)$, the reduced Compton wavelength (3) for m_b, and ω_1 from (2). The selected final state has $N+1$ elementary particles that eventually escape to large space-like separations with no excess energy. Then, the final energy-momenta are

$$(N+1)(\lambda_c^{-1}, 0, 0, 0).$$

Conservation of energy provides the rest mass of a bound state of N elementary particles. Interpreting the bound state as consisting of N of the single type of neutral, finite mass elementary particles, conservation of energy provides that

$$\sqrt{\lambda_b^{-2} + \mathbf{p}_1^2} + \omega_1 = (N+1)\lambda_c^{-1}$$

or

$$m_b = \left((N+1)^2 m^2 + m^2 - 2(N+1)\, m\, \frac{\hbar\omega_1}{c} \right)^{\frac{1}{2}}.$$

Then

$$0 \leq m_b \leq Nm$$

and the *binding energy* is designated as the rest mass energy difference of the N free particles and the bound complex. The greater the required $\omega_1 > m$, the more deeply bound are the N elementary particles. This scattering event exhibits no particle creations or annihilations.

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